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Hyperbolic Automorphisms and Holomorphic Motions in $\mathbb{C}^2$

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1. Introduction

Holomorphic motions have been an important tool in the study of complex dynamics in one variable. In this paper we provide one approach to using holomorphic motions in the study of complex dynamics in two variables. To introduce these ideas more fully, let Δ_r be the disk of radius r and center 0 in the plane, let $\mathbb{P}^1$ be the Riemann sphere, and recall that a holomorphic motion of a set $E \subset \mathbb{P}^1$ is a function $\alpha\colon \Delta_r \times E \to \mathbb{P}^1$ such that $\alpha(0, z) = z$ for each $z \in E$, $\alpha(\lambda, \cdot)\colon E \to \mathbb{P}^1$ is injective for each fixed $\lambda \in \Delta_r$, and $\alpha(\cdot, z)\colon \Delta_r \to \mathbb{P}^1$ is holomorphic for each fixed $z \in E$. For future reference, we note that this definition (as well as most results about holomorphic motions) applies equally well when the parameter λ is allowed to vary in the complex polydisk: $\lambda \in \Delta_r^n$.

One of the first uses of holomorphic motions in the study of complex dynamics was in [MSS], where holomorphic motions were used to prove the density of structurally stable maps within the family of polynomial maps of $\mathbb{C}$ of degree d. In general, a map $f\colon M \to M$, M a manifold, is structurally stable within a family $\mathcal{F}$ of maps if there is some neighborhood of f, say $\mathcal{U} \subset \mathcal{F}$, such that any map in $\mathcal{U}$ is conjugate to f via a homeomorphism of M. Mañé-Sad-Sullivan [MSS] obtained structural stability for polynomial maps by showing that (subject to certain restrictions) the holomorphic motion defined naturally on the Julia set of a polynomial map extends to give a conjugacy on all of $\mathbb{C}$ to nearby polynomial maps. More precisely, they did this by starting with the canonical holomorphic motion defined on hyperbolic periodic points and on periodic points satisfying a critical orbit relation; by their λ-lemma, this holomorphic motion extends uniquely to a holomorphic motion of the closure of the periodic points. The authors then constructed (by hand) certain holomorphic motions which give partial conjugacies and which extend by iteration to give a holomorphic motion of a dense set of the plane, which again extends uniquely to give a topological conjugacy on the whole sphere.

Shortly after this work, Bers and Royden [BeR] used the notion of a harmonic Beltrami coefficient (defined in Section 6) to show that, given any holomorphic motion of a set E, there is a canonical extension of this motion to a holomorphic

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