Michigan Math. J. 49 (2001)

Explicit Solutions to the H-Surface Equation on Tori

Henry C. Wente

I. Introduction

A twice continuously differentiable map x from $\Omega \subset \mathbb{R}^2$ into $\mathbb{R}^3$ is a solution to the H-surface equation if

$$\Delta x = 2H(x_u \wedge x_v) \text{ on } \Omega. \tag{1}$$

Here $\Delta x = x_{uu} + x_{vv}$ is the standard Laplacian on $\mathbb{R}^2$ and the wedge symbol denotes the usual cross product. We represent points in Ω by (u, v) or sometimes $w = u + iv$; points in the target space are represented by $x = (x, y, z) \in \mathbb{R}^3$.

It is important to observe that solutions to (1) remain solutions after a conformal change of coordinates. Thus it makes sense to consider solutions to the H-surface equations on a Riemann surface. This may be seen as follows. We set the Dirichlet integral to be

$$D(x) = \iint_\Omega (|x_u|^2 + |x_v|^2)\, du\, dv \tag{2}$$

and the oriented volume functional to be

$$V(x) = \frac{1}{3} \iint_\Omega x \cdot (x_u \wedge x_v)\, du\, dv. \tag{3}$$

Observe that $D(x)$ is invariant under a conformal change of coordinates and that $V(x)$ is a parametric integral invariant under any smooth change of coordinates. Solutions to (1) are extremals of the H-functional

$$E_H(x) \equiv D(x) + 4HV(x). \tag{4}$$

To any solution of (1) is attached the Hopf differential

$$(x_w \cdot x_w)dw^2 = F(w)dw^2. \tag{5}$$

This is a holomorphic quadradic differential ($F_{\bar{w}} = 0$) and in local coordinates $F(w) = (|x_u|^2 - |x_v|^2) - 2i(x_u \cdot x_v)$. A solution $x(u, v)$ of (1) represents a surface of constant mean curvature (cmc surface) only when $F(w) \equiv 0$. Solutions of the H-surface equation have the same relationship to cmc surfaces as harmonic maps to minimal surfaces.

In an earlier paper [6] and also more recently [7], the author took up the question of solutions to (1) on annular domains that vanish on the boundary. By conformal invariance one may assume that the domain is a standard annulus bounded

Received November 20, 2000. Revision received April 6, 2001.