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# On the Fundamental Groups at Infinity of the Moduli Spaces of Compact Riemann Surfaces

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## 1. Introduction and Statement of the Result

In his seminal manuscript *Esquisse d'un programme* (1984; now available in [LS]), Grothendieck explains that the structure of the tower of all moduli spaces of curves is somehow governed by its "first two levels" ("deux premiers étages"), that is, the moduli spaces of dimensions 1 and 2. We refer to the *Esquisse* and to [L] for more context and details about this statement. Let us only mention that, speaking in terms of topology, Grothendieck was concerned more precisely with the orbifold fundamental groups of the moduli spaces of curves; he explains that the above "principle" is essentially equivalent to the fact that the orbifold fundamental group of any moduli space of dimension $> 2$ is equal to its fundamental group at infinity. We do not recall here the notion of orbifold fundamental group, which is due to Thurston in a topological context, because we will be concerned only with the ordinary topological fundamental group—that is, the fundamental groups of the moduli spaces of curves viewed as manifolds, forgetting about their orbifold structure. In terms of analytic or algebraic geometry, this amounts to viewing them as coarse and not as fine moduli spaces for curves.

Before we consider moduli spaces of curves in detail, let us make precise the notion of fundamental group at infinity in a topological context. Note that it is less easy (although feasible) to do it in terms of algebraic geometry, because a quasiprojective variety cannot usually be exhausted by an increasing sequence of projective subvarieties; nor is it easy to define tubular neighborhoods of closed subvarieties. So let $M$ be a paracompact differentiable manifold and partially order the compact submanifolds (possibly with boundary) of $M$ by inclusion. Their complements define an obvious inverse system: if $K \subset K'$, we simply consider the inclusion $M \setminus K' \subset M \setminus K$. We need a base point for our fundamental group and exploit the fact that a fundamental group need not be based at a point but in fact at any simply connected subset of the ambient manifold. Here a *base point at infinity,* simply denoted by $*$, is given by an open part $U \subset M$ such that, for any compact set $K$, there exists a compact set $K'$ with $K \subset K'$ and $U \setminus K'$ nonempty and *simply* connected. Let $\pi_1$ denote, as usual, the topological fundamental group (functor).

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