Michigan Math. J. 49 (2001)

Dynamics of Polynomial Hamiltonian Vector Fields in $\mathbb{C}^{2k}$

Romain Dujardin

1. Introduction

The main purpose of this article is to provide a dynamical study of a natural class of holomorphic vector fields, namely polynomial Hamiltonian (or complex divergence-free) vector fields in $\mathbb{C}^2$. By "dynamical study" we mainly mean here "real-time study", in a situation where the complex orbits of the flow are well understood: these are only level sets of a polynomial in $\mathbb{C}^2$. We restrict ourselves to the polynomial case, which is relevant for many approximation problems (see Sections 4 and 5); this enables us to use the global geometry of level sets.

We also give a contribution to the study of Hamiltonian vector fields in $\mathbb{C}^{2k}$ $(k \geq 2)$, again by first studying polynomial fields.

We introduce now some terminology (see Section 6 for the higher-dimensional case). Let p be an entire function in $\mathbb{C}^2$ (with coordinates (z, w)). The holomorphic vector field

$$X_p = \left(\frac{\partial p}{\partial w}, -\frac{\partial p}{\partial z}\right) = \frac{\partial p}{\partial w}\frac{\partial}{\partial z} - \frac{\partial p}{\partial z}\frac{\partial}{\partial w}$$

is called the *Hamiltonian* vector field associated to (the Hamiltonian) p and the symplectic form $\omega = dz \wedge dw$. This terminology is justified, as in the real case, by the relation $i_{X_p}\omega = dp$. For further information, see [F] and [FS1]. One sees readily that the flow of X_p preserves each level set $\{p = c\}$. We also recall from [F] that the real-time flow of a holomorphic vector field has a holomorphic extension to a neighborhood in $\mathbb{C}$ of its domain in the real axis.

The outline of this paper is as follows. In Sections 2–4 we give a rather complete picture of the dynamics of a generic class of polynomial Hamiltonian vector fields in $\mathbb{C}^2$. Note that, in order to speak of generic properties, one needs to fix the degree. We hope this can be used as an example for further study.

We also prove that the "quasi-ergodic hypothesis" is satisfied for polynomial and entire Hamiltonian vector fields. This gives a new proof of a result of [FS3].

In Section 5, we use the preceding work to study exploding orbits of holomorphic Hamiltonian vector fields [FG1; FG2]. One says that an orbit *explodes* if it reaches infinity in finite time. The following theorem is due to Fornæss and Grellier [FG1].

Received March 6, 2000. Revision received April 19, 2001.