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Solving the d- and $\bar\partial$-Equations in Thin Tubes and Applications to Mappings

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1. The Results

Let $\mathbf{C}^n$ denote the complex n-dimensional Euclidean space with complex coordinates $z = (z_1, \dots, z_n)$. A compact $\mathcal{C}^k$-submanifold $M \subset \mathbf{C}^n$ $(k \ge 1)$, with or without boundary, is *totally real* if for each $z \in M$ the tangent space $T_z M$ (which is a real subspace of $T_z\mathbf{C}^n$) contains no complex line; equivalently, the complex subspace $T_z^C M = T_z M + iT_z M$ of $T_z\mathbf{C}^n$ has complex dimension $m = \dim_{\mathbf{R}} M$ for each $z \in M$. We denote by $\mathcal{T}_\delta M = \{z \in \mathbf{C}^n : d_M(z) < \delta\}$ the tube of radius $\delta > 0$ around M; here $|z|$ is the Euclidean norm of $z \in \mathbf{C}^n$ and $d_M(z) = \inf\{|z - w| : w \in M\}$.

For any open set $U \subset \mathbf{C}^n$ and integers $p, q \in \mathbf{Z}_+$ we denote by $\mathcal{C}^l_{p,q}(U)$ the space of differential forms of class $\mathcal{C}^l$ and of bidegree (p, q) on U. For each multi-index $\alpha \in Z_+^{2n}$ we denote by ∂^α the corresponding partial derivative of order $|\alpha|$ with respect to the underlying real coordinates on $\mathbf{C}^n$.

The following is one of the main results of the paper; for additional estimates see Theorem 3.1.

1.1. Theorem. *Let $M \subset \mathbf{C}^n$ be a closed, totally real, $\mathcal{C}^1$-submanifold and let $0 < c < 1$. Denote by $\mathcal{T}_\delta$ the tube of radius $\delta > 0$ around M. There is a $\delta_0 > 0$ and for each integer $l \ge 0$ a constant $C_l > 0$ such that the following hold for all $0 < \delta \le \delta_0$, $p \ge 0$, $q \ge 1$, and $l \ge 1$. For any $u \in \mathcal{C}^l_{p,q}(\mathcal{T}_\delta)$ with $\bar\partial u = 0$ there is a $v \in \mathcal{C}^l_{p,q-1}(\mathcal{T}_\delta)$ satisfying $\bar\partial v = u$ in $\mathcal{T}_{c\delta}$ and satisfying also the estimates*

$$\begin{aligned} \|v\|_{L^\infty(\mathcal{T}_{c\delta})} &\le C_0 \delta \|u\|_{L^\infty(\mathcal{T}_\delta)}; \\ \|\partial^\alpha v\|_{L^\infty(\mathcal{T}_{c\delta})} &\le C_l(\delta \|\partial^\alpha u\|_{L^\infty(\mathcal{T}_\delta)} + \delta^{1-|\alpha|}\|u\|_{L^\infty(\mathcal{T}_\delta)}), \quad |\alpha| \le l. \end{aligned} \tag{1.1}$$

If $q = 1$ and the equation $\bar\partial v = u$ has a solution $v_0 \in \mathcal{C}^{l+1}_{(p,0)}(\mathcal{T}_\delta)$, then there is a solution $v \in \mathcal{C}^{l+1}_{(p,0)}(\mathcal{T}_\delta)$ of $\bar\partial v = u$ on $\mathcal{T}_{c\delta}$ satisfying

$$\|\partial_j \partial^\alpha v\|_{L^\infty(\mathcal{T}_{c\delta})} \le C_{l+1}(\omega(\partial_j\partial^\alpha v_0, \delta) + \delta^{-l}\|u\|_{L^\infty(\mathcal{T}_\delta)})$$

for $1 \le j \le n$ and $|\alpha| = l$.

In the last estimate, $\omega(f, \delta) = \sup\{|f(x) - f(y)| : |x - y| \le \delta\}$ is the *modulus of continuity* of a function; when f is a differential form on $\mathbf{C}^n$, $\omega(f, t)$ is defined as

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