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Almost Periodicity and the Remainder in the Ellipsoid Problem

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1. Introduction

Let $\mathfrak{S} \in \mathbb{R}^{m\times m}$ $(m \geq 2)$ be a positive definite real matrix, let $Q[\mathfrak{x}] := {}^t\mathfrak{x}\mathfrak{S}\mathfrak{x}$ be the associated quadratic form, and let $Q^{-1}[\mathfrak{x}] := {}^t\mathfrak{x}\mathfrak{S}^{-1}\mathfrak{x}$. For $\mathfrak{a} \in \mathbb{R}^m$, define

$$N_{\mathfrak{a}}(x) := \#\{\mathfrak{x} \in \mathbb{Z}^m \mid Q[\mathfrak{x} - \mathfrak{a}] \leq x\}, \quad x \geq 1,$$

which is the number of lattice points in the ellipsoid $\mathfrak{a} + \sqrt{x}E$, where $E := \{\mathfrak{x} \in \mathbb{R}^m \mid Q[\mathfrak{x}] \leq 1\}$. A simple lattice point argument shows that

$$\Delta_{\mathfrak{a}}(x) := N_{\mathfrak{a}}(x) - \operatorname{vol}(E)x^{m/2} \ll x^{(m-1)/2},$$

where

$$\operatorname{vol}(E) = \frac{\pi^{m/2}}{(\det \mathfrak{S})^{1/2}\Gamma(m/2+1)}$$

is the Euclidean volume of E. Landau [18] improved this estimate to

$$\Delta_{\mathfrak{a}}(x) \ll x^{m/2-1+1/(m+1)} \quad (m \geq 2)$$

using the functional equation of the Epstein zeta function for Q. Krätzel and Nowak [17] derived (in the more general case of a convex body with smooth boundary of strictly positive Gaussian curvature) the better estimate $\Delta_{\mathfrak{a}}(x) \ll x^{m/2-1+\lambda}$ with

$$\lambda = \frac{5}{6m+2} \quad \text{for } m \geq 8, \qquad \lambda = \frac{12}{14m+8} \quad \text{for } 3 \leq m \leq 7.$$

They used exponential sum estimates. In the special case of a rational ellipsoid (i.e., when there is some $a > 0$ with $a\mathfrak{S} \in \mathbb{Q}^{m\times m}$), Landau [19] proved the estimate

$$\Delta_{\mathfrak{a}}(x) \ll x^{m/2-1} \quad (m \geq 5).$$

In this case the theory of theta series can be applied, giving better results. Recently the same estimate was proved by Bentkus and Götze [1] for an arbitrary real ellipsoid E and $m \geq 9$. For rational ellipsoids, the bound $O(x^{m/2-1})$ is optimal. For irrational ellipsoids and $m \geq 9$, Bentkus and Götze [2] showed that $\Delta_{\mathfrak{a}}(x) = o(x^{m/2-1})$, which has important applications to conjectures of Davenport and Lewis and of Oppenheim. In [2] the authors used techniques from probability theory that they originally invented to obtain optimal rates of convergence in central limit theorems.

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