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k-Plane Transforms and Related Operators on Radial Functions

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1. Introduction

In 1917, J. Radon proved that a smooth function in $\mathbb{R}^3$ is completely determined by its integrals over all the planes. This leads in a more general setting to consideration of the so-called k-plane transform. Let f be a smooth function in $\mathbb{R}^n$ and let $1 \le k < n$ be an integer. Denote by $G(n,k)$ the set (called the Grassmannian manifold) of all k-dimensional subspaces (or k-planes) of $\mathbb{R}^n$. The k-plane transform of f is defined as

$$Tf(x,\pi) = \int_\pi f(x-y)\,d\lambda_k(y)$$

for $x \in \mathbb{R}^n$ and $\pi \in G(n,k)$, where λ_k denotes the Lebesgue measure on π. When $k = 1$ this operator is usually named X-ray transform; when $k = n-1$, Radon transform. Such transformations have many practical and theoretical applications (see e.g. the references in [S]).

The properties of the k-plane transform depend on the properties of f. Here we are concerned with a size estimate measured in terms of a mixed norm inequality, namely,

$$\left(\int_{G(n,k)}\left(\int_{\pi^\perp}|Tf(x,\pi)|^q\,d\lambda_{n-k}(x)\right)^{r/q} d\gamma_{n,k}(\pi)\right)^{1/r} \le C_{p,q,r}\|f\|_p. \tag{1.1}$$

Here $\pi^\perp$ denotes the subspace orthogonal to π and $\gamma_{n,k}$ is the rotation-invariant measure on $G(n,k)$ (see [M, Chap. 3] for a construction of $\gamma_{n,k}$ and some of its properties). When inequality (1.1) holds for some p, the definition of the k-plane transform can be extended to $f \in L^p$ and $Tf(x,\pi)$ is finite for almost every translate of almost every k-plane.

A scaling argument replacing $f(x)$ by $f(\lambda x)$ shows that (1.1) is possible only if

$$\frac{n}{p} - \frac{n-k}{q} = k.$$

Moreover, checking the inequality against the characteristic function of a parallelepiped of sides $1 \times \delta \times \cdots \times \delta$, we can see that the condition

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