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How Far Is an Ultraflat Sequence of Unimodular Polynomials from Being Conjugate-Reciprocal?

Tamás Erdélyi

1. Introduction

Let D be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by ∂D. Let

$$\mathcal{K}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^{n} a_k z^k,\ a_k \in \mathbb{C},\ |a_k| = 1 \right\}.$$

The class $\mathcal{K}_n$ is often called the collection of all *complex* unimodular polynomials of degree n. Let

$$\mathcal{L}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^{n} a_k z^k,\ a_k \in \{-1, 1\} \right\}.$$

The class $\mathcal{L}_n$ is often called the collection of all *real* unimodular polynomials of degree n. By Parseval's formula,

$$\int_0^{2\pi} |P_n(e^{it})|^2 \, dt = 2\pi(n+1)$$

for all $P_n \in \mathcal{K}_n$. Therefore

$$\min_{z \in \partial D} |P_n(z)| \le \sqrt{n+1} \le \max_{z \in \partial D} |P_n(z)|. \tag{1.1}$$

An old problem (or rather an old theme) is the following.

Problem 1.1 (Littlewood's flatness problem). How close can a unimodular polynomial $P_n \in \mathcal{K}_n$ or $P_n \in \mathcal{L}_n$ come to satisfying

$$|P_n(z)| = \sqrt{n+1}, \quad z \in \partial D\,? \tag{1.2}$$

Obviously (1.2) is impossible if $n \ge 1$. So one must look for less than (1.2), but then there are various ways of seeking such an "approximate situation". One way is the following. Littlewood [Li1] suggested that there might conceivably exist a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ (possibly even $P_n \in \mathcal{L}_n$) such that

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