Michigan Math. J. 49 (2001)

Hausdorff Dimension and Limit Sets of Quasiconformal Groups

PETRA BONFERT-TAYLOR & EDWARD C. TAYLOR

1. Introduction

There is an extensive theory, due initially to Patterson and Sullivan, intertwining the isometric, conformal, and ergodic properties of Kleinian groups. Our purpose here is to begin to expand this theory to the setting of quasiconformal groups. In particular, we wish to explore the connection between the Hausdorff dimension of limit sets of quasiconformal groups and the exponent of convergence of the Poincaré series. It is well known that, for a large class of finitely generated Kleinian groups, the Hausdorff dimension of the limit set is the exponent of convergence [BiJo; S2]. We are concerned with the facet of Patterson–Sullivan theory that relates the exponent of convergence to the Hausdorff dimension of the limit set, and our techniques are primarily from the analytic theory of quasiconformal mappings. We will, however, directly apply techniques and results from Patterson–Sullivan theory in the sequel to this paper [BT].

The Poincaré series of a Kleinian group has been the object of much refined study; see, for example, [BiJo; Mc; Pa; S1; S2; Tu].

Because a quasiconformal group no longer acts isometrically on hyperbolic space, it is to be expected that the whole of Patterson–Sullivan theory does not generalize directly to quasiconformal groups. Thus our purpose in this paper is twofold: we record positive results and then provide counterexamples that demonstrate ways in which the Patterson–Sullivan theory fails for discrete quasiconformal groups.

The central part of the paper consists of Sections 4 and 5, where we provide examples to demonstrate differences between the quasiconformal and the conformal case. We find that *the exponent of convergence can be strictly greater than the Hausdorff dimension of the conical limit set* (Example 4.1) and that *the Hausdorff dimension can "jump up" in the limit on convergent sequences of quasiconformal groups* (see Example 4.2). For convergent sequences of Kleinian groups, the Hausdorff dimension of the limit set is lower semicontinuous (see [BiJo] and [Mc]). We also provide an example (Example 4.3) of a *discrete quasiconformal group whose limit set consists entirely of conical limit points*; *however, the group has the property that the Hausdorff measure of the limit set at the critical dimension has zero*

Received February 28, 2000. Revision received April 16, 2001.
The second author was supported in part by a NSF Postdoctoral Fellowship.