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A Szegö-Type Theorem for Gabor–Toeplitz Localization Operators

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1. Introduction and Preliminaries

Linear time-frequency methods such as Gabor expansions or the short-time Fourier transform provide ways of analyzing signals by describing their frequency content as it varies over time. In contrast to quadratic methods, Gabor expansions allow one to manipulate signals in a linear way. The resulting time-variant filtering procedures allow one to isolate partial signals or suppress the noise. Gabor expansions have a long history and have been applied by engineers for decades. Despite this fact, their mathematical properties still hide many intricate features. Their study has recently been substantially intensified ([AT; FS; QC; Te] are some recent books that deal with current developments).

The Gabor reproducing formula defines a natural context for time-frequency localization, which is one of the methods for time-variant filtering. By modifying by the weight function, the basic projection operators that build the reproducing formula, we emphasize those regions of the time-frequency plane that correspond to large values of the weight and disregard those where the weight is small. The resulting operator, assigning to the original signal its time-frequency modification, shares many features with classical Toeplitz operators and is called a *Gabor–Toeplitz localization operator*. The weight function is called a *symbol*.

One of the main lines of investigation of time-frequency localization operators is the study of the behavior of their eigenvalues as the domain of localization is expanded by dilations (see [Da2; LW; RT]). The distribution function, indicating how many eigenvalues are bigger than ε $(1 > \varepsilon > 0)$, is the principal object in this study. The distribution of the eigenvalues of Gabor–Toeplitz operators was first studied by Daubechies in [Da1]. She observed that if the symbol is a characteristic function of a disk centered at the origin and if the function defining the reproducing formula (i.e., the *window*) is a standard Gaussian, then the corresponding Gabor–Toeplitz localization operator is diagonalized by Hermite functions. She obtained a formula for the eigenvalues in terms of the restricted Gamma function, and this allowed her to analyze the behavior of the distribution of the eigenvalues. Later, a different approach (based on trace formulas) was taken by Ramanathan and Topiwala in [RT]. This approach enabled them to generalize several of the Daubechies results to the context of general bounded domains and general windows. In this

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