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F-Rational Rings and the Integral Closures of Ideals

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1. Introduction

The history of the Briançon–Skoda theorem and its ensuing avatars in commutative algebra have been well documented in many papers (see e.g. [AH1; LS]). We will therefore only briefly review the relevant concepts and theorems. First recall the definitions of the integral closure of an ideal.

DEFINITION 1.1. Let R be a ring and let I be an ideal of R. An element $x \in R$ is *integral over* I if x satisfies an equation of the form $x^n + a_1 x^{n-1} + \cdots + a_n = 0$, where $a_j \in I^j$ for $1 \leq j \leq n$. The *integral closure* of I, denoted by $\bar{I}$, is the set of all elements integral over I. This set is an ideal.

Let R^o be the set of all elements of R not in a minimal prime. An equivalent though less standard (but for our purposes a more useful) definition of integral closure is the following.

EQUIVALENT DEFINITION 1.1. Let R be a Noetherian ring and let I be an ideal of R. An element $x \in R$ is *integral over* I if there exists an element $c \in R^o$ such that $cx^n \in I^n$ for all $n \gg 0$.

A theorem proved by Briançon and Skoda [BS] for convergent power series over the complex numbers and generalized to arbitrary regular local rings by Lipman and Sathaye states as follows.

THEOREM 1.2 [BS; LS]. *Let R be a regular local ring and let I be an ideal generated by ℓ elements. Then, for all $n \geq \ell$,*

$$\overline{I^n} \subseteq I^{n-\ell+1}.$$

This was partially extended to the class of pseudo-rational rings by Lipman and Teissier [LT]. However, they were unable to recover the full strength of Theorem 1.2.

THEOREM 1.3 [LT, (2.2)]. *Let R be a Noetherian local ring and assume that the localization R_P is pseudo-rational for every prime ideal P in R. Suppose that I has a reduction J such that* dim $R_P \leq \delta$ *for every associated prime P of J^n. Then*

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