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The Structure of Pleasant Ideals

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Abstract Continuing the work begun in Pleasant Ideals (*Notre Dame Journal of Formal Logic* vol. 32 (1991) pp. 612–617), we investigate the relationships among selective, normal and pleasant ideals. Our major result is that any selective ideal extending NS_κ is normal.

1 Introduction and Preliminaries The investigation of normal ideals, ideals that are closed under diagonal unions, has been ongoing for many years. In our [2] we introduced the concept of a pleasant ideal, an ideal that is closed under diagonal unions indexed by members of the ideal. It seems that to ask an ideal to be pleasant is very close to asking it to be normal, in the sense that pleasantness combines with several other ideal properties to imply normality.

Our set theoretic notation is standard. The axiom of choice is assumed throughout so a cardinal is identified with the set of its ordinal predecessors. The letters κ and λ will be reserved for cardinals, while α, β, etc. will represent ordinals.

An ideal on a regular uncountable cardinal κ is a collection of subsets of κ that is closed under subset and finite union. Our ideals will contain all singletons and be $< \kappa$ complete, and thus will extend $I_\kappa \equiv \{X \subseteq \kappa \mid |X| < \kappa\}$. If I is an ideal on κ, then I^* will denote the the dual filter and I^+ will be the co-ideal $\{X \subseteq \kappa \mid X \notin I\}$. If I is an ideal and $A \in I^+$, then $I \restriction A$ is the ideal $\{X \subseteq \kappa \mid X \cap A \in I\}$.

If $A \subseteq \kappa$ and $f : A \rightarrow \kappa$, f will be called regressive if $f(\alpha) < \alpha$ for $\alpha \in A - \{0\}$, and weakly regressive if $f(\alpha) \leq \alpha$. If I is an ideal, then f is I-small if $f^{-1}(\{\xi\}) \in I$ for every $\xi < \kappa$.

The nonstationary ideal on κ, NS_κ, is defined by $A \in NS_\kappa \iff$ there is a club $C \subseteq \kappa$ such that $A \cap C = \varnothing$. It is well known that NS_κ is the smallest normal ideal, i.e., the smallest ideal that is closed under diagonal unions, so if $X_\alpha \in NS_\kappa$ for all $\alpha < \kappa$, then $\mathop{\nabla}\limits_{\alpha<\kappa} X_\alpha \equiv \{\xi < \kappa \mid (\exists \alpha < \xi)(\xi \in X_\alpha)\} \in NS_\kappa$. If I is any normal ideal and if $Q \in I^+$, then there is no I-small regressive function on Q.

An ideal I on κ is called a p-point if for any I-small $f : \kappa \rightarrow \kappa$ there exists a set $X \in I^*$ such that $f \restriction X$ is I_κ-small. I is a q-point if for any I_κ-small $f : \kappa \rightarrow \kappa$ there exists a set $X \in I^*$ such that $f \restriction X$ is one-to-one. I is selective if I is both a

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