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A GROUP-THEORETIC CHARACTERIZATION OF THE ORDINARY AND ISOTROPIC EUCLIDEAN PLANES*

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I. INTRODUCTION. It is known that every geometry determines a unique group, namely, the group of transformations under which the geometry remains invariant.

The converse problem is concerned with the determination of a geometry corresponding to an abstract group. The problem of characterizing a geometric space group-theoretically is solved by defining an abstract group in such a way that it determines uniquely the geometric space in question and such that its structure corresponds to that of the transformation group of the space.

Group-theoretic characterization of a geometric space is based on the line reflection as its fundamental concept. G. Hessenberg [5] and J. Hjelmslev [6, 7] first brought out the significance of the property—known as the theorem of the three reflections—that the product of the reflections in three lines concurrent in a point is again a reflection in a line through the same point. Hjelmslev used methods based on reflections systematically and studied the foundations of geometry in this light.

G. Thomsen [13] described the Euclidean plane group-theoretically and developed effective methods of proof based on reflection calculus. In his work Thomsen also indicated the extension of these methods to higher dimensional Euclidean spaces and to the projective and non-Euclidean planes.

A. Schmidt [10] assumed the theorem of the three reflections as an axiom. He was the first to formulate group-theoretic axioms for the plane absolute geometry, including the elliptic plane, giving dominance to the importance of the line reflections as generating the other motions. F. Bachmann [2] later reduced Schmidt's axioms and developed methods of considering metric planes for the Euclidean, hyperbolic and elliptic cases.

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