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ON PREDICATE LETTER FORMULAS WHICH HAVE NO SUBSTITUTION INSTANCES PROVABLE IN A FIRST ORDER LANGUAGE.

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We shall investigate the following question in this discussion. Does there exist an algorithm A which operates on a recursively enumerable formal system S couched in the first order predicate calculus P (say the formulas of *S* are constructed from logical symbols of P with predicate and individual symbols from given finite or infinite lists) such that if S is simple consistent, then A(S) is a satisfiable predicate letter formula which has no substitution instance provable in S? A partial solution is given in the theorem below. The notation used is from [1].

Theorem 1 (Kleene): For every recursively enumerable and simple consistent formal system S, *couched in the first order predicat calculus, there is a satisfiable formula* F *of* P *where* F *has no substitution instance provable in* S *and* F *can be effectively found, given* S.

The following proof is due to S. C. Kleene in [2]. We shall repeat the argument here, since [2] is not readily available.

Because S is recursively enumerable, we can enumerate recursively all the provable formulas of S. From each provable formula of S we can recover the finitely many formulas of P of which it is a substitution instance. Thus we can recursively enumerate the formulas of P which have substitution instances provable in S. Suppose the formulas of P in this enumeration are: $F_0, F_1, F_2, \ldots.$ Then

1) F_i is satisfiable $(i=0, 1,2, \ldots)$,

for if F_i were not satisfiable, then $\neg F_i$ would be valid and hence provable in P by Gödels completeness theorem. So if $F_i{}^*$ is any one of the substitution instances of F_i, which is provable in S, we would have $\neg F_i{}^*$ also provable and thus S is not simple consistent.

Consider the predicate $T_1(x,x,y)$ in [1, p. 281] and the formulas K_x in [1, p. 434, Remark 2] for $R(x,y) \equiv T_1(x,x,y)$.

2) $(y)\overline{T}_1(x,x,y) \equiv (\overline{Ey})T_1(x,x,y) \equiv [K_x \text{ is unprovable in P}]$
$\equiv [K_x \text{ is not valid}] \equiv [\neg K_x \text{ is satisfiable}]$

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