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CHURCH'S THEOREM ON THE DECISION PROBLEM[1]

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I. *Introduction*. This expository paper borrows from three main sources to present a proof of Church's theorem in the form

(1) *The set of valid quantificational formulas is not effective.*

A set C of formulas is called *effective* if there is a clerical routine which, for any given formula F, correctly answers Yes or No to the question whether $F \varepsilon C$. (Sets and relations of numbers will be called effective in parallel fashion.) But effectiveness, being an intuitive notion, must be replaced by a suitable formal analogue if we are to have a claim which admits of mathematical proof. The notion of *recursiveness* has been cast in this role, in what seems a paradigm of successful explication. Using this technical term and another soon to be explained, then, we arrive at

(2) *The set of Gödel numbers of valid quantificational formulas is not recursive.*

While (2) is provable, however, in order to infer (1) we need an additional premise, a version of Church's Thesis, namely,

(3) *All effective sets of numbers are recursive.*

But by using (3) and its analogue for relations, it turns out that a proof of (1) is possible which saves labor by not using (2) at all. This general strategy, whose source is Quine [6], will be the one followed here. The plan is (a) to exhibit a nonrecursive set of numbers, (b) to establish a link between recursiveness and quantificational validity, and then (c) in terms of this link, to show that the denial of (1) implies that the set presented in (a) is recursive. This proof, while owing its spirit to Quine, won't follow him in details, however. For phase (a), we borrow the same diagonal argument as Quine does from Kleene [3], but here adapt it to a definition of recursiveness, adapted from Smullyan [7], which lends itself particularly well to the purposes of phase (b).[2]

II. *Recursive enumerability and recursiveness* will be treated in terms of the *elementary arithmetic* (**EA**). An **EA** may be defined as a finite set of

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