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POSSIBILITY-ELIMINATION IN NATURAL DEDUCTION

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F. B. Fitch's extension of the subordinate-proof technique to modal logic[1] represents an interesting and valuable contribution to both study and exposition in the field. The modal introduction and elimination (intelim) rule-schemata he offers are these:

$\Box$**E**: $\begin{array}{|l} \Box p \\ \hline p \end{array}$ $\qquad$ $\Diamond$**I**: $\begin{array}{|l} p \\ \hline \Diamond p \end{array}$

$\Box$**I**: $\begin{array}{|l} \begin{array}{l|l} \Box & \vdots \\ & p \end{array} \\ \Box p \end{array}$ $\qquad$ $\Diamond$**E**: $\begin{array}{|l} \Diamond p \\ \hline \begin{array}{l|l} \Box & p \\ & \vdots \\ & q \end{array} \\ \Diamond q \end{array}$

$\sim\Box$**E**: $\begin{array}{|l} \sim\Box p \\ \hline \Diamond \sim p \end{array}$ $\qquad$ $\sim\Diamond$**I**: $\begin{array}{|l} \Box \sim p \\ \hline \sim \Diamond p \end{array}$

$\sim\Box$**I**: $\begin{array}{|l} \Diamond \sim p \\ \hline \sim\Box p \end{array}$ $\qquad$ $\sim\Diamond$**E**: $\begin{array}{|l} \sim \Diamond p \\ \hline \Box \sim p \end{array}$

If the propositional base, to which it is understood that these rules are appended, is classical, then a system similar to Lewis' S4 is obtained by permitting only propositions of the form $\Box p$ (or $\sim\Diamond p$) to be reiterated into the strict subordinate proofs of $\Box$**I** and $\Diamond$**E**. A weaker system similar to S2 is obtained by requiring such a reiterated proposition to drop its left-most modal operator.[2]

Two peculiarities, related in part to Fitch's restricted form of $\sim$**I**, emerge upon consideration of his modal rules. (1) Even on a classical base (which will be assumed throughout), the last four rules—those relating $\Box$ and $\Diamond$ —cannot be derived from the first four—the fundamental intelim rules for $\Box$ and $\Diamond$; and they are thus needed to complete the modal apparatus, (2) $\Box$**E** and $\Diamond$**I** can be derived from each other, and $\Diamond$**E** from $\Box$**I** (in the appropriate forms determined by the definition of $\Box p$ as $\sim\Diamond \sim p$ and $\Diamond p$ as $\sim \Box \sim p$). But $\Box$**I** in the form

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