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A NOTE ON THE GENERALIZED CONTINUUM HYPOTHESIS. I

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It is well-known[1] that the following set-theoretical formulas:

A *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$, *if* $\mathfrak{m} = \mathfrak{n}$, *then* $2^{\mathfrak{m}} = 2^{\mathfrak{n}}$

B *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$, *if* $2^{\mathfrak{m}} < 2^{\mathfrak{n}}$, *then* $\mathfrak{m} < \mathfrak{n}$

C *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$, *if* $\mathfrak{m} < \mathfrak{n}$, *then* $2^{\mathfrak{m}} < 2^{\mathfrak{n}}$

D *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$, *if* $2^{\mathfrak{m}} = 2^{\mathfrak{n}}$, *then* $\mathfrak{m} = \mathfrak{n}$

are such that a) A is provable without any difficulty in the general set theory,[2] b) we do not know whether it is possible to prove B without the help of the axiom of choice, and that c) we are unable to prove C and D without the aid of the generalized continuum hypothesis, i.e. the formula:

$\mathfrak{A}$ *If* $\mathfrak{m}$ *is a cardinal number which is not finite, then there exists no cardinal* $\mathfrak{n}$ *such that* $\mathfrak{m} < \mathfrak{n} < 2^{\mathfrak{m}}$

which, as we know,[3] is equivalent to

$\mathfrak{B}$ *The axiom of choice*

taken in conjunction with the formula

$\mathfrak{C}$ *for any ordinal number* α, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$

i.e. Cantor's hypothesis on alephs.

In this note I present several sets of assumptions such that

α) each of these sets is equivalent either to $\mathfrak{A}$ or to $\mathfrak{C}$.

β) almost each of these sets contains either formula C or formula D or a certain instance of one of these formulas.

In the considerations given below I am using constantly several of my results published in [6], and, especially, that formula $\mathfrak{C}$, i.e. Cantor's hypothesis on alephs, is equivalent to the conjunction of the following two formulas:

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