CERTAIN FORMULAS EQUIVALENT TO THE AXIOM OF CHOICE

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In this note it will be shown that each of the following four set-theoretical formulas:

$\mathfrak{A}$. *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\aleph(\mathfrak{m})$ *and* $\aleph(\mathfrak{n})$ *are the least Hartogs' alephs with respect to* $\mathfrak{m}$ *and* $\mathfrak{n}$ *respectively, and such that* $\aleph(\mathfrak{m}) = \aleph(\mathfrak{n})$, *then* $\mathfrak{m} = \mathfrak{n}$.

$\mathfrak{B}$. *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\aleph(\mathfrak{m})$ *and* $\aleph(\mathfrak{n})$ *are the least Hartogs' alephs with respect to* $\mathfrak{m}$ *and* $\mathfrak{n}$ *respectively, and such that* $\aleph(\mathfrak{m}) > \aleph(\mathfrak{n})$, *then* $\mathfrak{m} > \mathfrak{n}$.

$\mathfrak{C}$. *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\aleph(\mathfrak{m})$ *and* $\aleph(\mathfrak{n})$ *are the least Hartogs' alephs with respect to* $\mathfrak{m}$ *and* $\mathfrak{n}$ *respectively, and* $\mathfrak{m} > \mathfrak{n}$, *then* $\aleph(\mathfrak{m}) > \aleph(\mathfrak{n})$.

$\mathfrak{D}$. *For any cardinal numbers* $\mathfrak{m}$, $\mathfrak{n}$, *and* $\mathfrak{p}$, *if* $\mathfrak{m} < \mathfrak{p}$, $\mathfrak{n} < \mathfrak{p}$, *and for any cardinal numbers* $\mathfrak{x}$ *and* $\mathfrak{s}$, *if* $\mathfrak{m} + \mathfrak{x} = \mathfrak{p}$ *and* $\mathfrak{n} + \mathfrak{s} = \mathfrak{p}$, *then* $\mathfrak{x} = \mathfrak{s}$, *then either* $\mathfrak{m} \geqslant \mathfrak{n}$ *or* $\mathfrak{n} > \mathfrak{m}$.

is inferentially equivalent to the axiom of choice.[1]

Concerning these formulas it should be noted: 1) that the formulas $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ are related to the following theorem:

T1. *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\aleph(\mathfrak{m})$ *and* $\aleph(\mathfrak{n})$ *are the least Hartogs' alephs with respect to* $\mathfrak{m}$ *and* $\mathfrak{n}$ *respectively and* $\mathfrak{m} \leqslant \mathfrak{n}$, *then* $\aleph(\mathfrak{m}) \leqslant \aleph(\mathfrak{n})$.,

which is provable without the aid of the axiom of choice[2] and: 2) that the formula $\mathfrak{D}$ is a weaker formulation of the law of trichotomy for cardinal numbers, since in it the trichotomy is preceded by an antecedent consisting of three conditions.[3]

Proof:

(i) *The axiom of choice implies* $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, *and* $\mathfrak{D}$. It is easy to determine that these theorems follow from the axiom of choice. Viz., in order

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