FUNCTIONAL COMPLETENESS OF HENKIN'S PROPOSITIONAL FRAGMENTS

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It was shown in [1] that if $\phi(x_1, \ldots, x_2)$ has the defined property of being Tarskian, the addition of schemata $(\phi)*$ as in [2] to the positive logic of implication, *A1-2*, yields the complete system of classical implication. Knowledge of [1] is pre-supposed. We define:

Def. $\mathfrak{T}_1$ For all ϕ, ϕ is Tarskian$_1$ iff ϕ is Tarskian and is valued F when all its arguments are valued T.

Def. $\mathfrak{T}_2$ For all ϕ, ϕ is Tarskian$_2$ iff ϕ is Tarskian and is valued T when all its arguments are valued T.

THEOREM 1. If ϕ is Tarskian$_1$, $\{A1\text{-}2, (\phi)*\}$ is functionally complete.

Proof. If ϕ is Tarskian$_1$, the proof of Lemma 1, case (i)a and the corresponding sub-case of case (ii), in [1], shows that $\{A1\text{-}2, (\phi)*\}$ contains *S1-2* with each i-th argument of ϕ either A or $A \supset A$. Defining the negation of A for ϕ with these arguments, we get from *S1-2*:

$$(1)\quad A \supset . \sim A \supset C$$

$$(2)\quad A \supset C \supset . \sim A \supset C \supset C .$$

Taking C in (2) as A, and detaching $A \supset A$, we get

$$(3)\quad \sim A \supset A \supset A .$$

Since hypothetical syllogism is given by *A1-2*, and this with (1) and (3) constitutes the well known Łukasiewicz base for a full and functionally complete system in implication and negation, the theorem follows.

THEOREM 2. If ϕ is valued T when all its arguments are valued T, negation is not definable in the system $\{A1\text{-}3, (\phi)*\}$.

Proof. The system $\{A1\text{-}3, (\phi)*\}$ is, by [2], complete for tautologies in implication and ϕ. So every expression $A \supset B$ with A and B tautologous is provable, and by the hypothesis on ϕ, $\phi(A \supset A, \ldots, A \supset A)$ is provable. Hence every expression f (imp, ϕ, $A \supset A$) with implication and ϕ as the only functors, and all elementary argument places filled by $A \supset A$, is provable. We suppose now that negation is definable. We should have as provable

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