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NON-RECURSIVENESS OF THE SET OF FINITE SETS OF EQUATIONS WHOSE THEORIES ARE ONE BASED

DOUGLAS D. SMITH

Among the decision problems for finite sets of equations listed by A. Tarski [1] are six related problems, five of which were solved by Peter Perkins [2]. The solution for the one equation case of the remaining problem S_3, given below, follows closely the method (and notation) of Perkins, reducing the problem to the (unsolvable) word problem of a semigroup, but making a modification in a set of equations used by Perkins and introducing a transformation on terms which allows us to show that the equations do not "mix" (in the sense indicated by our lemma below). This property of "not mixing" was suggested by the work of W. E. Singletary on partial propositional calculi [3].

We assume the basic notions from [1]. Thus, for a set of equations $\mathbf{E}$, the equational theory of $\mathbf{E}$, $\mathsf{Th}(\mathbf{E})$, is one based iff there exists a single equation e such that $\mathsf{Th}(e) = \mathsf{Th}(\mathbf{E})$.

Theorem*. *The set of finite sets of equations whose equational theories are one based is not recursive. Specifically, there is no effective method for determining whether or not the equational theory of an arbitrary finite set of equations in two binary operation symbols and two constants is one based.*

Proof: Let β:$\{a, b; U_i = V_i, 1 \leq i \leq n\}$ be a finite presentation of a semigroup with unsolvable word problem. Let $\mathfrak{L}^+$ denote an equational language having one binary operation +. To each β-word (i.e., word in a and b) we make correspond a term $\mathsf{W}(x, y)$ in the language $\mathfrak{L}^+$, as follows:

if W is a, $\mathsf{W}(x, y)$ is $(y + x) + x$
if W is b, $\mathsf{W}(x, y)$ is $x + (x + y)$
if W is $a\mathsf{W}_1$, $\mathsf{W}(x, y)$ is $(\mathsf{W}_1(x, y) + x) + x$
if W is $b\mathsf{W}_1$, $\mathsf{W}(x, y)$ is $x + (x + \mathsf{W}_1(x, y))$.

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*In a letter received after our proof was completed, George F. McNulty indicates that he has a result from which ours follows.

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