Notre Dame Journal of Formal Logic
Volume XII, Number 4, October 1971
NDJFAM

CERTAIN COUNTEREXAMPLES TO THE CONSTRUCTION OF COMBINATORIAL DESIGNS ON INFINITE SETS

WILLIAM J. FRASCELLA

The present note attempts to elaborate the main result of my paper [1]. To this end the following definitions are necessary.*

Definition 1. Let M be some fixed set and F and G families of subsets of M. G is said to be a Steiner cover of F if and only if for every $x \in F$ there is exactly one $y \in G$ such that $x \subset y$.

Definition 2[1]. *Let k be a non-zero cardinal number such that $k \leq \overline{\overline{M}}$. A family F of subsets of M is called a k-tuple family of M if and only if* i) *if $x, y \in F$ such that $x \neq y$ then $x \not\subset y$ and* ii) *if $x \in F$ then $\overline{\overline{x}} = k$.*

As in [1] the result presented here will be given within Zermelo-Fraenkel set theory with the axiom of choice. If x is a set, $\overline{\overline{x}}$ denotes the cardinality of x. If n is a cardinal number then $[x]^{*n} = \{y \subset x : \overline{\overline{y}} * n\}$ where $*$ can stand for the symbols $=, \leq, \geq, <$ or $>$. The expression "$x \subset y$" means "x is a subset of y" improper inclusion not being excluded. If α is an ordinal number ω_α is the smallest ordinal whose cardinality is $\aleph_\alpha$. As usual, we write ω for ω_0. For each ordinal α we define a cardinal number $\mathfrak{a}_\alpha$ by recursion as follows: set $\mathfrak{a}_0 = \aleph_0$. If $\alpha = \beta + 1$ then set $\mathfrak{a}_\alpha = 2^{\mathfrak{a}_\beta}$. If α is a limit number then set $\mathfrak{a}_\alpha = \sum_{\beta<\alpha} \mathfrak{a}_\beta$. Also for any ordinal α, $\mathrm{cf}(\alpha)$ represents the smallest ordinal which is cofinal with α.

It is now possible to state the main result of [1] as follows.

Theorem 3. In every set M of cardinality $\mathfrak{a}_\omega$ there is an $\aleph_0$-tuple family F of M such that there does not exist a family $G \subset [M]^{\aleph_1}$ which is a Steiner cover of F.

The following will be the principal content of the present note.

Theorem 4. Let α, β and γ be ordinal numbers such that i) *$\alpha < \beta < \gamma$,* ii) *γ is a limit number,* iii) *$\mathrm{cf}(\omega_\gamma) \leq \omega_\alpha < \mathrm{cf}(\omega_\beta)$,* iv) *if $\delta < \gamma$ then $\aleph_\delta^{\aleph_\alpha} < \aleph_\gamma$ and*

*The present research was partially supported by the National Science Foundation under grant GP-14134.

Received March 3, 1969