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TWO NOTES ON VECTOR SPACES WITH RECURSIVE OPERATIONS

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In [1] the author studied an $\aleph_0$-dimensional vector space $\overline{U}_F$ over a countable field F; it consists of an infinite recursive set ε_F of numbers (i.e., non-negative integers), an operation $+$ from $\varepsilon_F \times \varepsilon_F$ into ε_F and an operation $\cdot$ from $F \times \varepsilon_F$ into ε_F. If the field F is identified with a recursive set, both $+$ and $\cdot$ are partial recursive functions. Let β be a subset of ε_F. We call β a *repère*, if it is linearly independent; β is an *α-repère*, if it is included in a r.e. repère. A subspace V of $\overline{U}_F$ is an *α-space*, if it has at least one *α-basis*, i.e., at least one basis which is also an α-repère. We write c for the cardinality of the continuum. It can be shown [1, pp. 367, 385, 386 and 2, §2] that among the c subspaces of $\overline{U}_F$ there are c which are α-spaces and c which are not. The present paper* contains improvements of two results obtained in [1]. Henceforth the notations and terminology of [1] will be used.

1. HAMILTON'S THEOREM. Every two α-bases of an *isolic* α-space are recursively equivalent. This result [1, p. 375, Corollary 2] was strengthened by A. G. Hamilton [2] to:

every two α-bases of any α-space are recursively equivalent.

This means that $\dim_\alpha V$ can be defined for any α-space V. The following proof is shorter than Hamilton's; it is a modification of the proof of T1 in [1].

Proof. Let β and γ be α-bases of the α-space V, say $\beta \subset \overline{\beta}$, $\gamma \subset \overline{\gamma}$, where $\overline{\beta}$ and $\overline{\gamma}$ are r.e. repères. If V is finite-dimensional we are done, hence we suppose that $\dim V = \aleph_0$; thus $\beta, \overline{\beta}, \gamma$ and $\overline{\gamma}$ are infinite sets. We have $V = L(\beta) = L(\gamma)$, $V \leq L(\overline{\beta})$, $V \leq L(\overline{\gamma})$. Note that $L(\overline{\beta})$ need not equal $L(\overline{\gamma})$. There is no loss of generality in assuming that $\overline{\beta} \subset L(\overline{\gamma})$. For suppose this were not the case; take $\beta_0 = \overline{\beta} \cap L(\overline{\gamma})$; then $\beta \subset \beta_0$, where β_0 is a r.e. repère included in $L(\overline{\gamma})$. Assume therefore that $\overline{\beta} \subset L(\overline{\gamma})$. Put $\gamma^* = \overline{\gamma} \cap L(\overline{\beta})$, then

$$\beta \subset \overline{\beta} \subset L(\overline{\gamma}),\ \gamma \subset \gamma^* \subset \overline{\gamma},\ \gamma^* \subset L(\overline{\beta}),$$

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