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INCOMPLETENESS VIA SIMPLE SETS

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Let P be Peano arithmetic and let Σ_0 be the set of formulas in the language of P which only contain bounded quantifiers. It is well known that if Q is an ω-consistent extension of P, and $Q(x)$ is a Σ_0-formula, then

(1) $Q \vdash (\exists x)\, \phi(x)$ *implies* $Q \vdash \phi(\mathsf{n})$ *for some* $n < \omega$.

What we show here is that by only slightly more complicating the form of ϕ, (1) will fail in every consistent axiomatizable extension of P.* In detail

Theorem: *There is a* Σ_0*-formula* $\phi(x, y, z)$ such that for any consistent *axiomatizable extension* Q *of* P *there is a* $q < \omega$ *such that* $Q \vdash (\exists x)(\forall y)\, \phi(x, y, \mathsf{q})$, *but for no* $n < \omega$ *does* $Q \vdash (\forall y)\, \phi(\mathsf{n}, y, \mathsf{q})$.

(Note that under these hypotheses (1) above implies our result is the best possible.)

Proof: Let S be the simple set of Post (*cf.* [1] p. 106). We define S in terms of the Kleene predicate T (which enumerates the n-th recursively enumerable set as $\{m : (\exists u)\, T(n, m, u)\}$), the pairing function j, and its first, second inverse k, l.

(2) $F(m, n) \equiv (\exists u)\,[(T(n, m, u) \wedge m > 2n) \wedge (\forall v)\,((v < j(m, u) \wedge T(n, k(v), l(v)) \rightarrow k(v) \leq 2n)]$

(3) $S(m) \equiv (\exists n)\, F(m, n)$

Let $\phi(y, x), \sigma(y)$ be the intuitive translations of F, S into the language of P and let Q be any consistent axiomatizable extension of P. F is a partial recursive function (in the n to m direction) which is represented in P (á fortiori Q) by

(4) $F(m, n)$ *implies* $Q \vdash \phi(\mathsf{m}, \mathsf{n})$,

and

(5) $Q \vdash (\phi(y, x) \wedge \phi(z, x)) \rightarrow y = z$.

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