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SUPERINDUCTIVE CLASSES IN CLASS-SET THEORY

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1. *Introduction*[1] Many definitions of the (von Neumann) ordinals have been given in set theory, but the one which seems most natural to us is that which parallels Frege's definition of the natural numbers, as the intersection of all inductive classes. This definition of ordinals as the intersection of all 'superinductive' classes has been proposed and its virtues discussed by Sion and Wilmot [3] and Smullyan [4]. In [4] a more general process of superinduction is discussed and the resulting minimally superinductive classes play a key role in particularly elegant proofs of Zorn's lemma, the Well-ordering theorem, and the Transfinite recursion theorem. Methods of establishing the existence of this minimally superinductive class in versions of Class-Set theory such as Gödel's [2], where we cannot assert the existence of classes defined by formulas containing bound class variables, have been briefly described in [4]. In Smullyan [5] a proof is given of the existence of the minimally superinductive class in Gödel's Class-Set theory which though proving a slightly more general theorem than the one we present here, requires both the axiom of substitution and the axiom of choice. In section 2, we present a new proof which avoids using the axiom of substitution and the axiom of choice. In addition as a by-product of our proof we obtain yet another definition of ordinal and a new definition of constructible set. In section 4, we present a proof that the minimally superinductive class under an arbitrary progressing function is well-ordered under inclusion. This theorem is given in [4] for slowly progressing functions. Again our proofs avoid using the axiom of substitution.

2. *Existence of Minimally Superinductive Classes*

Definition. A function g is called *progressing* if $x \subseteq g(x)$ for all x in the domain of g.

Definition. A function g is *slowly progressing* if g is progressing and $g(x)$ contains at most one more element than x.

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1. The results presented here are contained in the first chapter of the author's thesis [1], written under the supervision of Professor Raymond Smullyan at Yeshiva University.

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