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GENERALIZED REALS

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Real numbers fill the line, thanks to a postulational dictum—the axiom of completeness of the line. But since dimension depends on structure and not on cardinality, such an axiom is justified only by the simplicity it yields. In principle, there is no limit to the number of points that can be fitted on the line.

We shall assume that standard reals have already been introduced and shall refer to them as reals of the first kind, or simply reals; their set will be designated R. To the field R we now apply the operation of ultraproduct with R itself as an index set and with ultrafilter U containing all the co-countable subsets of R as well as all the upper halves of R (all sets of reals greater than, or greater than or equal to, a given r in R). The ultraproduct R_2 thus obtained is the quotient field of the set of all functions of R into R—denoted by A—modulo U. Members of R_2 will be called reals of the second kind, or r-reals. R_2 is a totally ordered non-Archimedean field containing infinites and infinitesimals. The monad $\mu(r)$ of a r-real r is the set of elements of R_2 infinitely close to r. For definitions see [3, p. 57].

Theorem 1. *R_2 is of cardinality $2^c = c_2$, which is also the cardinality of any monad.*

Proof: Because the cardinality of R_2 is the cardinality of any of its monads [2, p. 200], it suffices to show that R_2 has cardinality c_2. To do this, consider any one-to-one function f on R into R. The set P_f of f's permutations is of cardinality c_2 [4, p. 193]. We wish to show that there is a subset of P_f, also of cardinality c_2, composed exclusively of functions which are pairwise different modulo U. This will prove the theorem, since the cardinality of P_f/U is less than or equal to the cardinality of $A/U = R_2$. Let us consider, then, all chains (ordered by inclusion) of members of U in the Boolean algebra of all subsets of R. There must be c_2 of these chains because each is of cardinality c and the cardinality of U is c_2. There is no minimal element in any of these chains (otherwise the empty set would be in U). If for any permutations f_1 and f_2 of f, $f_1 = f_2$ (mod U), then there is a set u in U such that for all ξ in u, $f_1(\xi) = f_2(\xi)$. Also, for every η in every subset v of u in U, $f_1(\eta) = f_2(\eta)$. Therefore, as we move along a chain of

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