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ON A PROBLEM OF TH. SKOLEM

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1. Introduction. As pointed out in [2] the standard definition of an ordered pair, viz. $\langle x,y\rangle = \{\{x\}, \{x,y\}\}$, does not generalize in a natural way to ordered n-tuples. For example, the candidate $\langle x_1, x_2, x_3\rangle = \{\{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}\}$ is no good since this gives $\langle x, y, y\rangle = \langle x, x, y\rangle$. The standard generalization to n-tuples is given by $\langle x_1\rangle = x_1$, $\langle x_1, \ldots, x_{n+1}\rangle = \langle\langle x_1, \ldots, x_n\rangle, x_{n+1}\rangle$. However, this definition has the unusual property that every n-tuple is also an m-tuple for $2 \leq m \leq n$. Also if x_1, x_2, x_3 are of type k in simple type theorem, then $\langle x_1, x_2\rangle$ is of type $k + 2$, hence $\langle x_1, x_2, x_3\rangle = \langle\langle x_1, x_2\rangle, x_3\rangle$ is not type-theoretically well-defined.

The generalizations proposed in [2] are rather awkward in form. In this paper we offer several solutions to Skolem's problem of finding a "best" definition for ordered n-tuples. The idea is to start with some new definitions of "ordered pair" which in turn do generalize in several natural ways, the "best" choice depending upon what conditions we wish ordered n-tuples to satisfy. Some possible conditions are as follows:

(C1) $\langle x_1, \ldots, x_n\rangle = \langle y_1, \ldots, y_n\rangle \Longrightarrow x_i = y_i$ for $1 \leq i \leq n$;
(C2) all n-tuples ($n \geq 2$) are actually 2-tuples;
(C3) $m \neq n \Longrightarrow \langle x_1, \ldots, x_m\rangle \neq \langle y_1, \ldots, y_n\rangle$;
(C4) in simple type theory, if $x_1, \ldots, x_n$ are of the same type, then $\langle x_1, \ldots, x_n\rangle$ is well-defined.

Of course we want all definitions to satisfy C1. Conditions C2 and C3 are clearly mutually exclusive. C2 is a property possessed by the standard definition of ordered n-tuples, whereas C3 is closer to the intuitive notion of n-tuples. Condition C4 was considered in [2].

Let T_0 be a pure set or set-class theory satisfying the axioms of extensionality and pair set, $T_1 = T_0$ + null set axiom, and $T_2 = T_1$ + adjoining set axiom ($x, y \in V \Longrightarrow x \cup \{y\} \in V$). Small Roman letters denote set variables. Finally, let $x^{[0]} = x$, $x^{[n+1]} = \{x^{[n]}\}$ for $n \geq 0$.

2. First Definition. Consider the basic definition $\langle x,y\rangle = \{\{\emptyset, x\}, \{y\}\}$ which trivially satisfies C1 for case $n = 2$. Several possible generalizations are now defined.

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