Notre Dame Journal of Formal Logic
Volume XI, Number 3, July 1970

NOTE ON G. J. MASSEY'S CLOSURE–ALGEBRAIC OPERATION

BOLESŁAW SOBOCIŃSKI

1. In [1] it was shown by Massey that a binary operation defined as follows:

Df1 $A * B =_{Df}[-(-A \cap *A \cap -*B) \cup A] \cap [(-A \cap *A \cap -*B) \cup -(A \cap B)]$

where $*$, $\cap$, $\cup$ and $-$ are the symbols of closure, intersection, union and complementation operations respectively is functionally complete in closure algebras in the same sense that an operation of nonunion $(-(A \cup B))$ is functionally complete in Boolean algebras. In order to prove this Massey used a well known fact that closure algebras are in some sense strictly related to system S4 of Lewis, and, therefore, Df1 corresponds to the following definition in the latter system:

Df2 $A * B =_{Df}(\sim A . \Diamond A . \sim\Diamond B \supset A) . [\sim(\sim A . \Diamond A . \sim\Diamond B) \supset \sim(A . B)]$

Subsequently, using Kripke's semantics for S4 he has proved that the functor defined in Df2 can be adopted as a single primitive term of the modal system S4. Hence, operation $*$ defined in Df1 also possesses the required properties.

2. Below, using elementary algebraic calculations I shall show that in the field of closure algebras Df1 is inferentially equivalent to a much shorter formula, and, subsequently, starting from this new formula I shall prove algebraically the results which in [1] are obtained semantically. Moreover, it will be shown that a formula due to Massey in which the definability of intersection by operation $*$ is established can be substituted by a shorter one. An acquaintance with Boolean and closure algebras, as also with paper [1], is presupposed. Instead of $*$ the more common C will be used as a symbol of closure operation, and 0 and 1 will mean algebraic zero and unit elements. In the proof lines the calculations obtained by Boolean algebra will be indicated simply by **BA**. From closure algebras only the following theses will be used:

C1 $[a b]: a \epsilon A . b \epsilon A . \supset . \mathrm{C}(a \cup b) = \mathrm{C}a \cup \mathrm{C}b$
C2 $[a]: a \epsilon A . \supset . a \leqslant \mathrm{C}a$
C3 $[a]: a \epsilon A . \supset . \mathrm{C}\mathrm{C}a = \mathrm{C}a$
C4 $\mathrm{C}0 = 0$

Received February 5, 1970