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A NOTE ON A LEMMA OF J. W. ADDISON

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In [2] J. W. Addison, under the assumption of the axiom of constructibility, proved the following proposition:[1]

(C^1) *There exists an ω_1 well-ordering $<$ of N^N such that for any subset C of N^N and any predicate R recursive in functions in C, the set $\hat{\alpha}\hat{\beta}(E\beta_1)_{\beta_1<\beta}$ $(E\alpha)(x)R(\alpha,\beta_1,\alpha,x)$ is in $\Sigma_2^1[C]\cap\Pi_2^1[C]$.*

In his proof of (C^1) Addison used $V = L$ only in the proof of the following lemma (1.3). If we define:

(1.0) $W(\phi) \equiv \phi(i,j) = 0$ well orders N,
(1.1) ϕ_i = the ordinal number corresponding to i in the well-ordering $\phi(i,j) = 0$,
(1.2) $M(\phi,\varepsilon) \equiv W(\phi) \,\&\, \varepsilon(i,j) = 0 \equiv F'\phi_i \epsilon F'\phi_j$,

and if we let $<$ be the ω_1 well-ordering of N^N defined by

$\alpha < \beta$ if and only if the least ordinal ν such that $\omega\times\omega\cdot F'\nu = \alpha$ is less than the first ordinal μ such that $\omega\times\omega\cdot F'\mu = \beta$,

then we have:

$$\begin{aligned} L(1.3)\quad & (E\beta_1)_{\beta_1<\beta}(E\alpha)(x)R(\alpha,\beta_1,\alpha,x) \\ & \equiv (E\beta_1)[(E\phi)(E\varepsilon)[M(\phi,\varepsilon)\,\&\,(Ei)[\omega\times\omega\cdot F'\phi_i = \beta_1] \\ & \quad \&\sim(Ei)[\omega\times\omega\cdot F'\phi_i = \beta]]\,\&\,(E\alpha)(x)R(\alpha,\beta_1,\alpha,x)] \\ & \equiv (\phi)(\varepsilon)[M(\phi,\varepsilon)\,\&\,(Ei)[\omega\times\omega\cdot F'\phi_i = \beta]\rightarrow \\ & \quad (E\beta_1)[\beta_1\neq\beta\,\&\,(Ei)[\omega\times\omega\cdot F'\phi_i = \beta_1] \\ & \quad \&(E\alpha)(x)R(\alpha,\beta_1,\alpha,x)]]. \end{aligned}$$

We will show that L(1.3) can be proved under the weaker assumption that all real numbers are constructible ($N^N \subset L$) and that in fact L(1.3) is equivalent to $N^N \subset L$.[2] Thus we have the weakest assumption under which Addison's method can be used to prove (C^1).

Theorem $N^N \subset L \equiv$ *there exists an ω_1 well-ordering $<$ of N^N such that* L(1.3) *holds.*

Proof: The last two formulas of L(1.3) are equivalent by logic, so it

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