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A NEW PROOF OF THE COMPACTNESS THEOREM FOR PROPOSITIONAL LOGIC

ROBERT H. COWEN

The compactness theorem for propositional logic states that a denumerable set of propositional formulas is satisfiable if every finite subset is satisfiable. Though there are many different proofs, the underlying combinatorial basis of most of them seems to be König's lemma on infinite trees (see Smullyan [2], Thomson [3]). We base our proof on a different combinatorial lemma due to R. Rado [1], which allows us to easily prove a more general compactness theorem, viz., a well-ordered set of propositional formulas is satisfiable if every finite subset is (one gets a non-denumerable set of formulas by allowing non-denumerably many propositional variables).

Lemma (R. Rado). *Let I be well-ordered and for each $v \epsilon I$, let A_v be a finite set. For any finite $N \subset I$, suppose there is a function x_N on N satisfying $x_N(v) \epsilon A_v$ for $v \epsilon N$. Then there are elements x_v^*, such that given any finite N, there exists finite $N' \supseteq N$ with $x_v^* = x_{N'}(v)$, for $v \epsilon N$.*

Generalized Compactness Theorem. Let $S_1, S_2, \ldots, S_v, \ldots$ be a well-ordered sequence of propositional formulas. Then the entire set is satisfiable if every finite subset is.

Proof. Let I be the set of indices of the S_v. For each $v \epsilon I$, let A_v be the (finite) set of truth functional assignments to the variables of S_v, which satisfy S_v. Given any finite $N \subset I$, there is, by hypothesis, an assignment to the variables of the S_v, $v \epsilon N$, which simultaneously satisfies all S_v for $v \epsilon N$. Let $x_N(v)$ be the restriction of this assignment to S_v. Clearly $x_N(v) \epsilon A_v$.

Rado's lemma gives us assignments $x_v^* \epsilon A_v$ and we claim the x_v^* determine one assignment which simultaneously satisfies all the S_v. It is only necessary to show that for all v, $v' \epsilon I$, x_v^*, $x_{v'}^*$ agree on their common variables. By Rado's lemma, there exists $N' \supset \{v, v'\}$ such that $x_{N'}(v) = x_v^*$, $x_{N'}(v') = x_{v'}^*$ and hence x_v^*, $x_{v'}^*$ are restrictions of the same assignment and so must agree on their common variables.

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