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A PROOF OF THE LÖWENHEIM-SKOLEM THEOREM

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A modification of the proof procedure described in the appendix to Quine's *Methods of Logic* (revised edition) yields a simple proof[1] of a strong version of the Löwenheim-Skolem Theorem:

Any interpretation has an elementarily equivalent subinterpretation whose universe is countable.

(**I** is a subinterpretation of **J** if the universe of **I** is a subset of that of **J**, **I** assigns any (free) variable the same denotation as **J**, and **I** assigns any predicate letter the restriction to its own universe of whatever **J** assigns it. Two interpretations are elementarily equivalent if the same schemata are true under both.)

Proof. Let **J** be an interpretation, S_1, S_2, S_3, . . . an enumeration of all and only the prenex schemata true under **J**, and v_1, v_2, v_3, . . . an infinite non-repeating sequence of variables[2] none of which occurs in any S_i. We suppose all bound variables in the S_is to have been relettered so that no variable occurs bound in some S_i and free in some S_j. We further suppose S_1 to begin with an existential quantifier. We form an (infinite) list of schemata by writing down S_1, then applying EI as many times as we can, then applying UI as many times as we can, then writing down S_2, then applying EI as many times as we can, then applying UI as many times as we can, then writing down S_3, then applying . . . . In forming the list, we never

1. Our proof of the strong version stated above proceeds by a route somewhat different from that taken by the classical proofs, [2] and [3], in that we avoid the use of "Skolem normal forms" for satisfiability, and do not (apparently) close a countable (= finite or countably infinite) set under all Skolem functions for all prenex sentences true under the original interpretation. The method we have used can be extended to give a proof of the sill stronger version of the theorem stated in [3]: Any interpretation has an elementary subinterpretation whose universe is countable.

2. Each v_i is alphabetically later than any variable in any S_j; if $n > m$, v_n is alphabetically later than v_m. *Cf.* [1], p. 164.

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