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SEMI-BOOLEAN LATTICES*

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An implicative semi-lattice is an algebraic system $\langle L, \leq, \wedge, *\rangle$ in which $\langle L, \leq, \wedge\rangle$ is a meet semi-lattice, and $*$ is a binary composition such that $x \leq y * z$ if and only if $x \wedge y \leq z$ for all elements x, y, z, of L. Every implicative semi-lattice has a greatest element, denoted by 1. If an implicative semi-lattice has a least element 0, then it is called bounded. In a bounded implicative semi-lattice L, elements of the form $x * 0$ are called "closed". The set of closed elements forms a Boolean algebra which is a sub-implicative semi-lattice of L but not necessarily a sub-lattice. By a sub-lattice of an implicative semi-lattice we shall mean a sub-implicative semi-lattice which is a lattice and such that the join of any two elements of the sub-lattice is also a join in the semi-lattice.

An implicative lattice is simply an implicative semi-lattice which happens to be a lattice. Birkhoff [1] identifies bounded implicative lattices with Brouwerian logics. In general, the join of an implicative lattice is not very closely related to the implication. An exception to this is the case of a Boolean algebra. Here the join of two elements a and b always equals the element $(a * b) * b$. With this as a starting point, we make the following definition.

Definition 1. By the *pseudo-join* ab, of two elements a and b of an implicative semi-lattice L, we shall mean the element $((a * b) * b) \wedge ((b * a) * a)$.

Theorem 1. Let L be an implicative semi-lattice, and let a and b be elements of L. Then

(1) $a \leq ab$, $b \leq ab$
(2) $a \leq b$ *if and only if* $ab = b$
(3) $aa = a$, $ab = ba$
(4) $a(a \wedge b) = a = a \wedge (ab)$.

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