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ON EXPLANATION OF NUMBER PROGRESSION

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In a recent article[1] Benacerraf asserts that the less-than relation R must be recursive in order for the set of elements on which R is defined to constitute a set of numbers. Benacerraf thinks that this is an essential and independent requirement for a set of elements to be a set of numbers. He rejects Quine's view that there is only one condition upon all acceptable explications of numbers, namely, a given set of elements must be a progression in the sense that it is an infinite series each of whose members has only finitely many precursors.[2] Now the question is whether the condition that R must be recursive is an independent one for the characterization of natural numbers. In this paper I shall show that a little closer examination of Quine's view should dispel the doubt that the condition in question is not an independent one.

For our purpose, it suffices to show that the less-than relation R can be defined in terms of Quine's characterization of natural numbers as a progression, and that the recursiveness of the less-than relation is an inherent feature of the progression by definition. Quine defines[3] the class of natural numbers N as follows:

$$\mathrm{N} \equiv_{df} \{x\colon (z)\ (x \epsilon z . \breve{\mathrm{S}}``z \subseteq z . \supset . 0 \epsilon z)\}$$

In this sense to be a natural number is to be a member of all classes z fulfilling the initial condition "$0 \epsilon z$" and the closure condition "$\breve{\mathrm{S}}``z \epsilon z$" (precursors of $z$ are in $z$). A class $z$ which fulfills the initial condition "$0 \epsilon z$" and the closure condition "$\breve{\mathrm{S}}``z \epsilon z$" is in fact a progression each of whose members has only finitely many precursors. Of course, the definition does not restrict the size of a natural number, for a natural number can be infinitely large if there is a z which is infinitely large, even though an axiom of infinity for z is not needed to make sense of the definition.

To see that a progression is a class z, let the progression be:

$$A = a_1, a_2, a_3, \ldots, a_n, \ldots$$

Since each a_i has only finitely many precursors, each a_i must have 0 as its precursor, and has all the elements which are the precursors of each of its precursors as precursors. Thus A is indeed a class z.

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