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THE CHURCH ROSSER THEOREM FOR STRONG REDUCTION IN COMBINATORY LOGIC

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The Church Rosser theorem concerns a property relating to certain preordering relations [2a]. Originally it was stated for lambda conversions in a paper by Church and Rosser [1].

To define the property let Γ be a preordering and = be its symmetric closure. The property in question states

(C R). *If $M = N$, then there is an L such that $M \ \Gamma \ L$ and $N \ \Gamma \ L$.*

In this paper we give a proof that strong reduction (as modified by the author in a previous paper [3]) has the property **(C R)**. For strong reduction, the symmetric closure is simply combinatory logic with equality [2b]. The following results were proved in [3] and [5] and will be used here.

Lemma 1. If $X = [x]\mathfrak{X}$, then $\lambda x.Xx \succ\!\!- \lambda x.\mathfrak{X}$ by Type I steps only. In other words, the contractum of a Type III step may be reversed to the original redex by a single Type II step followed by Type I steps.

Lemma 2. The contraction of a Type II redex P may be reversed provided there are no intervening steps interior to the contractum of P.

Lemma 3. (Theorem 2.II of [5]) *If there is a standard reduction from M_0 to M_n and if there is a single step of Type I or III from M_0 to N_0, then there is an N_n such that there is a standard reduction $N_0 \succ\!\!- N_n$ and M_n.*

Lemma 4. (Lemma 5 of [5]) *If there is a reduction from M_0N_0 beginning with a Type II step yielding $(\lambda x.M_1)N_0$ and continuing to $(\lambda x.M_m)N_n$, then there is a reduction from M_0N_0 to $[N_n/x]M_m$ (where $[N_n/x]M_m$ means the substitution of N_n for x in M_m) in exactly the same number of steps.*

Lemma 5. (Theorem 3 of [5]) *If there is a strong reduction from X to Y where neither X nor Y contain lambda expressions, then there is a Z such that there is a standard reduction from X to Z and $Y \succ\!\!- Z$.*

Lemma 6. (Corollary A of [5]) *If there is a strong reduction from M to N, then there is a Z such that there is a reduction consisting of zero or*

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