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ARITHMETIC OPERATIONS ON ORDINALS

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1 *Introduction** We characterize addition and multiplication of ordinal numbers. We assume familiarity with the basic properties of ordinal arithmetic (Sierpiński [3], Chapter 14). Although our discussion is informal, it could be formalized within Gödel-Bernays set theory, e.g., within the axiom system consisting of groups A, B, C, and D of Gödel [1].

Greek letters, sometimes with subscripts, will denote ordinals; "On" will denote the class of all ordinals. As usual, "+" and "·" stand for ordinal addition and multiplication, respectively. Braces will designate proper classes as well as sets.

2 *Addition* Let + be any binary operation on On that is such that for all ordinals α, β, and γ,

1) $\alpha + 0 = \alpha$;
2) if $\beta \leqslant \gamma$, then $\alpha + \beta \leqslant \alpha + \gamma$;
3) if $\beta \leqslant \gamma$, then there is a unique δ such that $\beta + \delta = \gamma$.

In Proposition 2.1 and its corollary, we assume that + is a binary operation on On that satisfies 1), 2), and 3).

Proposition 2.1 *Let α, β, and γ be ordinals. If $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.*

Proof: $\alpha = \alpha + 0 \leqslant \alpha + \gamma$, by 1) and 2). Thus, if $\alpha + \beta = \alpha + \gamma$, then by 3), $\beta = \gamma$. By 2), $\alpha + \beta \leqslant \alpha + \gamma$; therefore, we must have $\alpha + \beta < \alpha + \gamma$.

Corollary *For all ordinals α, β, and γ, $\beta < \gamma$ if and only if $\alpha + \beta < \alpha + \gamma$.*

Define $+_1$, $+_2$, and $+_3$ on On as follows:

For $\alpha, \beta \in \mathrm{On}$,

$$\alpha +_1 \beta = \beta;$$
$$\alpha +_2 0 = \alpha,$$

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