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SUBTRACTIVE ABELIAN GROUPS

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1 *Introduction* In an earlier paper of the author [1], the operation of subtraction was used to develop a simple set of axioms for Boolean algebras. It became apparent that the requirement of a distinguished element 1 and of the axioms

K1 $(a - b) - c = (a - c) - b$
K2 $1 - (1 - a) = a$
K3 $a - a = 1 - 1$
K4 $a - (a - b) = a - (1 - b)$

defines a Boolean algebra if one sets $a' = 1 - a$, $a \wedge b = a - b'$, and $a \vee b = (a' - b')$.

Thomas M. Hearne and Carl G. Wagner [2], also using subtraction, have defined a generalized Boolean algebra by means of only three axioms including K1. As the axioms K1-K3 are valid in every additive group with the usual definition of $a - b$ and with an arbitrary choice of the distinguished element 1, the question arises which further axiom has to be added to these three axioms to define an Abelian group. It suffices, in fact, to replace K4 by $a - (a - b) = b$. This obviates axiom K2 and, together with K1, even implies axiom K3. Let, therefore, a subtractive group be a system $\langle S, -\rangle$ which satisfies the axioms

S1 $(a - b) - c = (a - c) - b$
S2 $a - (a - b) = b$.

It is easy to show that these two axioms are satisfied in any Abelian group. It is the purpose of this paper to show that S1 and S2 may also be used to define an Abelian group.

Let us first demonstrate the independence and consistency of these axioms. Consider the set $S = \{a, b\}$ and let $a - a = a - b = b - b = a$, $b - a = b$. Then S1 holds, but not S2, for $a - (a - b) = a - a = a \neq b$. If instead we put $x - y = y$ for all $x, y \in S$, axiom S2 is satisfied, but not S1, since $(a - a) - b = b$ while $(a - b) - a = a$. If, finally, we set $a - a = b - b = a$ and $a - b = b - a = b$ both axioms are satisfied. This shows their consistency.

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