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SUMS OF α-SPACES

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1 *Introduction** In [1] and [2], Dekker introduced and studied an $\aleph_0$-dimensional recursive vector space $\overline{U}_F$ over a countable field F. Briefly, it consists of an infinite recursive set ϵ_F of numbers (i.e., non-negative integers), an operation $+$ from $\epsilon_F \times \epsilon_F$ into ϵ_F and an operation $\cdot$ from $F \times \epsilon_F$ into ϵ_F. If the field F is identified with a recursive set, both $+$ and $\cdot$ are partial recursive functions. Let β be a subset of ϵ_F. We call β a *repère* if it is linearly independent; β is a *r.e. repère* if β is a r.e. set; and β is an *α-repère* if it is included in some r.e. repère. A subspace V of $\overline{U}_F$ is an *α-space* if it has at least one *α-basis*, i.e., at least one basis which is also an α-repère. A subspace V is *isolic* if it includes no r.e. repère; it is *r.e.* if it is r.e. as a set. The word "space" is used in the sense of "subspace of $\overline{U}_F$", and we denote "W is a subspace of V" by "$W \leqslant V$". We usually write (0) for $\{0\}$, and $\overline{U}$ for $\overline{U}_F$. Let $\alpha \subset \epsilon_F$. If $\alpha = \emptyset$, $\mathsf{L}(\alpha) = (0)$. If $\alpha \neq \emptyset$, $\mathsf{L}(\alpha)$ denotes the span of α, i.e., the set of all linear combinations (with coefficients in F) of finitely many elements of α. If $\alpha = \{a_0, \ldots\}$, we usually write $\mathsf{L}(a_0, \ldots)$ instead of $\mathsf{L}(\{a_0, \ldots\})$. We use $\mathfrak{c}$ to denote the cardinality of the continuum.

The reperes β and γ are *independent* if they are disjoint and their union is a repère. The spaces V and W are *independent* if $V \cap W = (0)$. The sets β and γ are *separable* (written: $\beta|\gamma$) if they can be separated by r.e. sets. The α-repères β and γ are *α-independent* (written: $\beta||\gamma$), if they can be separated by independent r.e. repères. The spaces V and W are *α-independent* (written: $V||W$), if there are independent r.e. spaces $\overline{V}$ and $\overline{W}$ such that $V \leqslant \overline{V}$ and $W \leqslant \overline{W}$. For spaces V, W, W is an *α-subspace* of V (written: $W \leqslant_\alpha V$) if there is an α-space S such that $W||S$ and $W \oplus S = V$.

In [3] we proved that the intersection of two α-spaces need not be an α-space. The same question naturally arises concerning the sum of two

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