

## A SUBSTITUTION PROPERTY

ALAN C. WILDE

Let  $(X, +, \cdot)$  be a commutative ring with identity 1, and let  $X' = \{x \in X \mid x^2 = x\}$ . Also, let  $x \cup y = x + y - xy$ ,  $x \cap y = xy$ , and  $\bar{x} = 1 - x$ . Then it is known that  $\langle X', \cup, \cap, - \rangle$  is a Boolean algebra. We want to explore a property of a function  $u: (X')^m \rightarrow X'$  and a function  $h: (X')^n \rightarrow X'$  that are of the form  $u(x_1, \dots, x_m) = \sum a_{i_1 \dots i_m} x_1^{(i_1)} \dots x_m^{(i_m)}$  and  $h(y_1, \dots, y_n) = \sum b_{i_1 \dots i_n} y_1^{(i_1)} \dots y_n^{(i_n)}$  where  $i_k, a_{i_1 \dots i_m}, b_{i_1 \dots i_n} \in \{0, 1\}$ ,  $x_i^{(1)} = x_i$ , and  $x_i^{(0)} = \bar{x}_i$  (i.e., disjunctive normal form). If we let  $v(x_i) = u(x_1, \dots, x_i, \dots, x_m)$  with all variables constant except  $x_i$ , we can prove the following:

**Theorem**  $v(h(y_1, \dots, y_n)) = v(1)v(0) \prod_{k=1}^n v(y_k) + v(1)\overline{v(0)} \sum b_{i_1 \dots i_n} \prod_{k=1}^n v^{(i_k)}(y_k) + \overline{v(1)}v(0) \sum \overline{b_{i_1 \dots i_n}} \prod_{k=1}^n v^{(1-i_k)}(y_k).$  (1)

*Proof:* We verify equation (1) by truth-value analysis, and then the Stone-Representation Theorem shows that (1) holds in a Boolean algebra. Substituting truth values  $j_1, \dots, j_n$  for  $y_1, \dots, y_n$ , we see that all terms vanish except those in which  $i_k = j_k$  ( $1 \leq k \leq n$ ). Thus it reduces to

$$v(h(j_1, \dots, j_n)) = v(1)v(0) + b_{j_1 \dots j_n} v(1)\overline{v(0)} + \overline{b_{j_1 \dots j_n}} \overline{v(1)}v(0). \quad (*)$$

If  $b_{j_1 \dots j_n} = 1$ , the RHS<sup>1</sup> of (\*) becomes  $v(1)v(0) + v(1)\overline{v(0)} = v(1)$ ; if  $b_{j_1 \dots j_n} = 0$ , the RHS of (\*) equals  $v(1)v(0) + \overline{v(1)}v(0) = v(0)$ . But the LHS of (\*) equals  $v(b_{j_1 \dots j_n})$  in any case. Thus we did the first part, and we are done.

Since  $v(1)v(0)\overline{v(y)} = 0$  and  $\overline{v(1)}v(0)v(y) = 0$  (by truth-value analysis and the Stone-Representation Theorem), terms of the form  $v^{(i)}(1)v^{(i)}(0) \prod_{k=1}^n v^{(i_k)}(y_k)$  where one  $i_k$  does not equal  $i$  also equal 0. The question occurs as to what are all the forms of  $v(h(y_1, \dots, y_n))$  that are linear combinations of conjunctions containing  $v(1), v(0), v(y_1), \dots, v(y_n)$ . All conjunctions are of the kind we discussed above and also of the form

---

1. The abbreviations RHS and LHS stand for "right-hand side" and "left-hand side" respectively.