Notre Dame Journal of Formal Logic
Volume XV, Number 3, July 1974
NDJFAM

GENERALIZATIONS OF THE DISTRIBUTIVE AND ASSOCIATIVE LAWS

ALAN C. WILDE

1 Introduction Let $x \triangle y$ and $x \bigcirc y$ denote two truth-value functions: $\{0, 1\} \times \{0, 1\} \rightarrow \{0, 1\}$, where 1 and 0 denote "true" and "false" respectively. The two functions "and" and "or" satisfy the law

$$(*) \qquad x \triangle (y \bigcirc z) = (x \triangle y) \bigcirc (x \triangle z)$$

in either order. We would like to weaken (*) so that more functions satisfy the relationship. To do so, we use

$$(**) \qquad x \triangle (y \bigcirc z) = (x \triangle y) \bigcirc (x \triangle z) \bigcirc (x \triangle \mathrm{I})$$

where I is the identity of $x \bigcirc y$. (**) is a generalization of (*) for the reason that all functions $x \bigcirc y$ that have identities and all $x \triangle y$ that together satisfy (*) also satisfy (**), but not conversely. This is shown in Theorem 1.

"And" and "or" satisfy the associative law

$$x \triangle (y \triangle z) = (x \triangle y) \triangle z,$$

and so does "equivalence" and "exclusive or." However, we shall demonstrate that for all truth-functions $x \triangle y$, the truth-values of $x \triangle (x \triangle z) \equiv (x \triangle y) \triangle z$ and $x \triangle (y \triangle z) \underline{\vee} (x \triangle y) \triangle z$ are independent of y.

2 The Generalized Distributive Law We wish to prove the following:

Theorem 1 (**) *holds*

(a) *for all* $x \triangle y$ *if* $x \bigcirc y$ *is either* $x \equiv y$ *or* $x \underline{\vee} y$;

and

(b) *for all* $x \triangle y$ *such that* $y \leqslant z$ *implies* $x \triangle y \leqslant x \triangle z$ *if* $x \bigcirc y$ *is* $x \vee y$ *or* $x \wedge y$.

Proof: Note that $x \wedge y$, $x \vee y$, $x \equiv y$, and $x \underline{\vee} y$ are the only functions that have identities, so Theorem 1 has all the possible combinations. All four of them happen to be commutative and associative. For part (a), let us show

Received September 18, 1972