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GENTZEN SYSTEMS FOR MODAL LOGIC

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Nice Gentzen formulations of the normal modal systems T and S4 have been know for some time; see, for example, Kanger [5] or Curry [1]. A similar formulation of S5 has also been given, but it is not so nice as the Elimination Theorem is not provable for it. I shall present here sets of rules for several of the non-normal modal systems which are akin to T and S4. Each of the L-systems to be defined here has an Elimination Theorem which may be proved by the methods of Gentzen [3]. These systems are useful, in that each of them has a decision procedure, following, for example, Kleene [6], §80.

1 Epistemic Systems In order to provide a decision procedure for Lewis' system S2, Ohnishi and Matsumoto [11] defined a system, Q2, which had the property that a formula, A, was provable in S2 if and only if the consecution $\mathsf{N}(p \supset p) \Vdash A$ was provable in Q2. Q2 was formed by adjoining to any appropriate formulation of the classical propositional calculus, such as Gentzen's **LK** [3], the rules:

$$(\mathsf{N}\Vdash)\ \frac{\Gamma, A \Vdash \Delta}{\Gamma, \mathsf{N}A \Vdash \Delta} \quad \text{and} \quad (\Vdash\mathsf{N})\ \frac{\Gamma \Vdash A}{\mathsf{N}\Gamma \Vdash \mathsf{N}A}$$

where for $(\Vdash\mathsf{N})\,\Gamma$ must be non-empty.

Here, and elsewhere, A, B, C, etc. are well formed formulas formed from atomic formulas by means of propositional connectives, including N for necessity; Γ, Δ, etc. are any finite sequences of (zero or more) constituent formulas, A, B, C, etc. Consecutions α, β, etc. are expressions $\Gamma \Vdash \Delta$. $\mathsf{N}\Gamma$ is the result of applying the operator N to each member of Γ.

Q2 is equivalent to the Hilbert style system E2 introduced by Lemmon (see, for example, [8]), in the sense that A is provable in E2 if and only if $\Vdash A$ is provable in Q2.

Besides the axioms and rules for the classical propositional calculus, E2 has only the axioms

A.1 $\mathsf{N}A \supset A$
A.2 $\mathsf{N}(A \supset B) \supset .\, \mathsf{N}A \supset \mathsf{N}B$

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