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BOOLEAN SUBTRACTIVE ALGEBRAS

THOMAS M. HEARNE and CARL G. WAGNER

1 Introduction In a recent paper [2], R. Güting has investigated structures $\langle K, -\rangle$, called colonies, which possess a distinguished element 1 and satisfy the following axioms:

K1 $(a - b) - c = (a - c) - b$
K2 $1 - (1 - a) = a$
K3 $a - a = 1 - 1$
K4 $a - (a - b) = a - (1 - b)$.

Güting shows that the study of such structures is equivalent to the study of Boolean algebras in the sense that every colony $\langle K, -\rangle$ gives rise to a Boolean algebra $\langle K, \vee, \wedge, '\rangle$ via the definitions $a' = 1 - a$, $a \wedge b = a - b'$, and $a \vee b = (a' - b)'$, and every Boolean algebra $\langle K, \vee, \wedge, '\rangle$ gives rise to a colony via the definition $a - b = a \wedge b'$.

In the present paper, we consider structures $\langle S, -\rangle$ which satisfy

S1 $(a - b) - c = (a - c) - b$
S2 $a - (b - a) = a$
S3 $\forall a, b \in S, \exists x \in S$ *such that* $x - (a - b) = b$ *and* $x - (b - a) = a$,

and prove that the study of such structures is equivalent to the study of generalized Boolean algebras. We call such structures Boolean subtractive algebras since they are subtractive algebras in the sense of Crapo and Rota ([1], 3.7). Alternatively, such structures might be called generalized colonies since, as we later prove, every colony is a Boolean subtractive algebra.

As an example of a Boolean subtractive algebra which is not a colony, we mention the set of all finite subsets of an infinite set, with set difference as composition. This (infinite) model shows the consistency of our axioms. There are also many finite models of S1, S2, and S3, all of which are colonies.

The independence of these axioms is also easily demonstrated. Let S be any two-element set and let $x - y = x$ for all $x, y \in S$. This composition shows the independence of S3. If, on the other hand, one sets $x - y = y$ for

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