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$\mathcal{K}$ AND $\mathcal{Z}$

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Goldblatt has shown in [1] that a system belongs to Sobociński's family $\mathcal{Z}$ if and only if it is the intersection of S5 and a certain member of the family $\mathcal{K}$. Offered here is an alternative proof of this same result which I obtained independently a short while after the appearance of the $\mathcal{Z}$ systems. The strategy is very similar to that used by Goldblatt, but there are enough differences of detail so that the present argument may still be of some interest.

We begin by establishing that if

$$\beta = CLMApqCLMNqCMKApqNqLMKApqNq$$
$$\gamma = CLMApNqCLMqCMKApNqqLMKApNqq$$
$$\xi = CLMCqLqCLMCNqLNqCMKCqLqCNqLNqLMKCqLqCNqLNq,$$

then $CK\xi K\gamma\beta ALCMpLMpLCLMqMLq$ is a thesis of S4. Suppose not. Then there exists a reflexive and transitive Kripke-style model $\mathfrak{A} = \langle w, W, R\rangle$ and valuation V on $\mathfrak{A}$ such that

$$\mathsf{V}(CK\xi K\gamma\beta ALCMpLMpLCLMqMLq, w) = 0,$$

whence

$$\mathsf{V}(\xi, w) = 1 \tag{1}$$
$$\mathsf{V}(K\gamma\beta, w) = 1 \tag{2}$$
$$\mathsf{V}(LCMpLMp, w) = 0 \tag{3}$$
$$\mathsf{V}(LCLMqMLq, w) = 0. \tag{4}$$

There are now two cases to be considered.

Case 1. Suppose

$$\mathsf{V}(MALqLNq, w) = 1,$$

whence it follows that there is some $x \in W$ such that wRx and either $\mathsf{V}(Lq, x) = 1$ or $\mathsf{V}(LNq, x) = 1$. But then $\mathsf{V}(KCqLqCNqLNq, x) = 1$ and therefore

$$\mathsf{V}(MKCqLqCNqLNq, w) = 1. \tag{5}$$

Moreover, from (4) it follows that there is some $y \in W$ such that wRy and

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