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A NOTE ON FINITE INTERMEDIATE LOGICS

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By an *intermediate propositional calculus* (**IPC**) we mean a propositional calculus whose formulae, or synonymously, sentences are constructed in the standard way from a denumerable set of propositional variables and connectives $\rightarrow$, $\wedge$, $\vee$, $\neg$, and whose theorems are determined by the single rule modus ponens and the axiom schemes for Heyting's (intuitionistic) propositional calculus together with a single extra axiom scheme which is a two-valued tautology. We denote Heyting's propositional calculus by **HC** and the **IPC** with 'extra' axiom scheme X by **HC** $+ X$. Two **IPC** are said to be *equivalent* if they have identical sets of theorems. (Up to equivalence we lose no generality in specifying that an **IPC** has just a single 'extra' axiom scheme from the case where an arbitrary finite number are allowed since a propositional calculus with axiom schemes those of **HC** together with, say, $X_1, \ldots, X_n$ is equivalent to **HC** $+ X_1 \wedge X_2 \wedge \ldots \wedge X_n$.)

We use the term *model* (often called matrix) of a propositional calculus in an entirely conventional sense, and since every model of an **IPC** is a fortiori a model of **HC** the study of models of **IPC**'s can be interpreted as the study of Heyting algebras, also called pseudo-Boolean algebras, which are pseudo-complemented lattices with smallest element. There is an account in [4]. A model is said to be *characteristic* for an **IPC** if a sentence is a theorem of the **IPC** if and only if it is valid in the model. An **IPC** which has a finite characteristic model will be called *finite*.

The purpose of this paper is to describe an effective test to determine of an arbitrary **IPC** whether it is equivalent to a given finite **IPC**. The test will be of a particular sort: associated with a finite **IPC** **HC** $+ U$, with characteristic model $\mathfrak{M}$, will be a finite set of finite models, and an arbitrary **IPC** **HC** $+ X$, will be equivalent to **HC** $+ U$ if and only if (i) X is valid in $\mathfrak{M}$, and (ii) X is invalid in each of the associated finite models. As an example of such a result in a particular case we have that **HC** $+ X$ is equivalent to **HC** $+ P \vee \neg P$, classical propositional calculus, if and only if X is a two-valued tautology and is invalid in the model which, considered as a lattice, has three linearly ordered elements: a result of Jankov.

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