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INCIDENCE RINGS OF PRE-ORDERED SETS

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Introduction. In this paper* every relation $\leqq$ on a set X is a binary relation which is transitive and reflexive. G. C. Rota [2] has defined incidence rings of partially ordered systems $\langle X, \leqq\rangle$. We generalize these rings by dropping the anti-symmetric condition on the order $\leqq$.

If X is a set and $\leqq$ a binary relation on X, then $\langle X, \leqq\rangle$ shall denote this relational system. We say that $\langle X, \leqq\rangle$ is a pre-ordered relational system if the relation $\leqq$ is transitive and reflexive. If confusion is unlikely, then we shall often take the liberty of using the relation $\leqq$ to denote the usual ordering of the natural numbers and also to denote a relation on a set X. Unless otherwise stated 0,1 should be understood to be real numbers. To each relational system $\langle X, \leqq\rangle$ there is a unique *zeta function*, ζ, mapping $X \times X$ into $\{0,1\}$. For $x, y \epsilon X$, $\zeta(x,y) = 1$, if $x \leqq y$ and $\zeta(x,y) = 0$ otherwise. In the context of a relation system $\langle X, \leqq\rangle$, $[x,y] = \{u \epsilon X \mid x \leqq u \leqq y\}$ is an *interval* and $\langle X, \leqq\rangle$ is *locally finite* iff every such interval is empty or a finite set.

We shall consider only rings R which have a multiplicative identity; rings may or may not be commutative. We do not assume any relationship between the rings R and sets X we discuss. The symbol R^* denotes the set of units of the ring R; the function det is the determinant function. If n is a positive integer, then $\mathsf{M}(n, R)$ denotes the complete ring of $n \times n$ matrices over the ring R. If X is any set, then $\mathcal{S}_X$ denotes the group of permutations of the set X; for positive integers n, $\mathcal{S}_n$ denotes $\mathcal{S}_{\{1,\ldots,n\}}$.

For a given ring R and locally finite pre-ordered system $\langle X, \leqq\rangle$, the *incidence ring* $I = \langle X, \leqq, R\rangle$ is set—theoretically the set of functions f mapping $X \times X$ into R satisfying the following *order condition.* For every $x, y \epsilon X$, $f(x,y) \neq 0$ only if $x \leqq y$. Multiplication, addition and scalar multiplication for incidence rings are defined in section 1. If $[x,y]$ is a

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