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A FORMAL CHARACTERIZATION OF ORDINAL NUMBERS

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In this paper we present the axioms for a first-order finitely axiomatized theory **ORD**, some of whose models are relational systems $\mathcal{S}$ with the following particular characteristics:

(i) S, the domain of discourse of $\mathcal{S}$, is any ordinal number;

and

(ii) each primitive relation symbol of the alphabet of **ORD** is interpreted in $\mathcal{S}$ in the standard manner.

Of special importance is the fact, demonstrated below, that **ORD** is an example of a theory in which the proof-theoretic notions of explicit and implicit definability, as stated in Beth [1], [2] and Smullyan [3], may be illustrated.

1 Basic Concepts. Let T be a first-order theory whose non-logical axioms are the set of sentences denoted by Γ_0. Let P, P_1, P_2 . . . be the relation symbols of the alphabet of T which occur in at least one member of Γ_0. In addition, P will be assumed to be an n-place relation symbol for some positive integer n.

P is *explicitly definable* from P_1, P_2 . . . in T if there exists a wff $U(x_1, x_2, \ldots, x_n)$, all of whose relation symbols occur in the list P_1, P_2 . . ., such that

$$\Gamma_0 \vdash (\forall x_1)(\forall x_2) \ldots (\forall x_n) [P(x_1, x_2, \ldots, x_n) \rightleftarrows U(x_1, x_2, \ldots, x_n)].$$

Let P' be a relation symbol of the alphabet of T having the same number of places as P. Assume P' does not occur in Γ_0, and let Γ_0' be the result of substituting P' for P in every sentence of Γ_0 in which P appears.

P is *implicitly definable* from P_1, P_2 . . . in T if

$$\Gamma_0 \cup \Gamma_0' \vdash (\forall x_1)(\forall x_2) \ldots (\forall x_n) [P(x_1, x_2, \ldots, x_n) \rightleftarrows P'(x_1, x_2, \ldots, x_n)].$$

2 The Theory **ORD**. The first-order theory **ORD** is, basically, a theory with equality, such that the four binary relation symbols, $\approx$, $\subset$, $\subseteq$, and ϵ

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