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INCOMPLETENESS THEOREM VIA WEAK DEFINABILITY OF TRUTH: A SHORT PROOF

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Introduction According to Tarski [9] no consistent theory T in which all recursive functions are definable allows definability of the truth of its own sentences, in the sense that there exists no formula Θ (with at most one free variable) such that for all sentences Φ

$$(\Theta(\mathbf{D}_{g(\Phi)}) \leftrightarrow \Phi) \in T,$$

where $g(\Phi)$ is the Gödel-number of Φ and $\mathbf{D}_{g(\Phi)}$ is the digit representing it. We shall refer to the definability of truth in this sense as "strong definability of truth." Myhill [8] defines a system S which allows definability of its own truth in the sense that there is a Θ such that for all Φ

$$\Theta(\mathbf{D}_{g(\Phi)}) \in S \textit{ iff } \Phi \in S.$$

We shall refer to the definability of truth in this sense as "weak definability of truth," because if truth is definable in the first sense, it follows that truth is definable in the later one.

In [2] Germano, solving a problem which in [1] Germano has left open, proves that the weak truth definability is a property of every recursively enumerable arithmetic in which all recursive functions are definable.

In [3] Germano gives a strong formulation of the incompleteness theorem (concerning every recursively enumerable arithmetic in which the elementary functions, i.e., the functions of the class $\mathcal{E}^2$ of Grzegorczyk [5], are definable) by comparing opportune formulations of the theorem on weak definability of truth and of the theorem on strong definability of truth. The present note gives a proof of the incompleteness theorem in the same strong formulation as in Germano [3], using only the theorem on weak definability of truth as announced in Germano [4]. The proof obtained in this way is the shortest direct proof of the incompleteness theorem known to the author and it is characterized by the fact that it mirrors step by step the construction of the *liar*'s paradox, which will be discussed later. A future paper will treat the possibility of extending the incompleteness

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