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A NEW REPRESENTATION OF S5

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We consider first a modal language with propositional constants (and no variables) and show that there is a unique set H of formulas of this language meeting certain attractive syntactical conditions; moreover H is the set of theses of a very simple calculus. We then show that the theses of S5 are characterized by the fact that all their instances are in H.*

Let $\mathcal{L}_c$ be the language having an infinite set of "propositional constants" and connectives $\neg$, $\vee$, and $\Box$ used in the usual way. As usual, other connectives are used as abbreviations. If S is a string of symbols, $s_1, \ldots, s_n$ are distinct symbols, and $S_1, \ldots, S_n$ are strings of symbols, then $S(S_1, \ldots, S_n/s_1, \ldots, s_n)$ is the result of replacing each symbol $s_i (i = 1, \ldots, n)$ in S by the string S_i. A *tautology* is a string of the form $X(S_1, \ldots, S_n/x_1, \ldots, x_n)$ where X is a tautology of the classical propositional calculus and $x_1, \ldots, x_n$ are propositional variables. A set H of formulas of $\mathcal{L}_c$ is *correct* if for all formulas A and B of $\mathcal{L}_c$

(1) If A is a tautology then $A \in H$.
(2) If A has no occurrences of $\Box$ and $A \in H$, then A is a tautology.
(3) If $A \in H$ and $A \Rightarrow B \in H$, then $B \in H$.
(4) $A \in H$ if and only if $\Box A \in H$.
(5) Either $A \in H$ or $\neg\Box A \in H$.

Let $\mathcal{L}_v$ be the language which is like $\mathcal{L}_c$ except that $\mathcal{L}_v$ has a countably infinite set of "propositional variables" rather than propositional constants. A set J of formulas of $\mathcal{L}_v$ is said to be correct if it consists of all formulas X of $\mathcal{L}_v$ such that every formula of $\mathcal{L}_c$ of the form $X(A_1, \ldots, A_n/x_1, \ldots, x_n)$ is a member of H, where H is a correct set of formulas of $\mathcal{L}_c$.

Let $\mathfrak{C}$ be the formal system whose language is $\mathcal{L}_c$, whose axioms are an appropriate set of tautologies and all formulas of the form

$$\Diamond \,\&\, \{a_i^* \mid i = 1, \ldots, n\}$$

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