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ON THE EXTENSIONS OF S5

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1 The purpose of this paper is to investigate the extensions of the Lewis system S5. To some extent this is meant to be a complement to what is shown in Scroggs [7]. We will show that any formula containing only one variable, if added to S5, will give an inconsistency or make the system collapse into classical propositional calculus (**PC**). We then examine the proper extensions of S5 obtained by adding formulas containing more than one variable. We describe Kripke-type semantics for these systems and prove their completeness.

2 A *normal* extension of S5 is an extension which is closed under the rules of substitution and (material) detachment. A *proper* extension of S5 is a normal extension where some formula not valid in S5 is derivable, but the formula $p \rightarrow Lp$ is not derivable.

Theorem 1: *If any wff, containing only one propositional variable, is added as a new axiom to* S5, *then the system thus obtained is not a proper extension of* S5.

Proof: Any formula is, in S5, equivalent to a formula in modal conjunctive normal form (**MNCF**).[1] We want to show that no formula of the form

(i) $$a \vee Lb_1 \vee Lb_2 \vee \ldots \vee Lb_n \vee Mc$$

where $a, b_1, \ldots b_n$ and c all are **PC**-formulas (possibly empty), can be used to give a proper extension of S5. Since any **MNCF**-formula is a conjunction of disjunctions of this form the theorem will follow. Our assumption is that $a, b_1, \ldots b_n$ and c contain only one propositional variable, say p. In [3] it is shown[2] that a formula of the form (i) is S5-valid (and hence derivable) iff at least one of the formulas $a \vee c$, $b_1 \vee c, \ldots, b_n \vee c$ is **PC**-valid. So in the added formula, none of these

1. See Hughes and Cresswell [3], pp. 54-56.

2. See Hughes and Cresswell [3], pp. 118-120.

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