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A NOTE ON OMITTING THE REPLACEMENT SCHEMA

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In [1] Heath considers a formalisation of primitive recursive arithmetic similar to that given in Goodstein [2], in which the replacement schema (Goodstein's $\mathbf{Sb}_2$) is deduced from special cases of itself, using a double recursive uniqueness rule. The deduction of $\mathbf{Sb}_2$ given in [1] is, however, incomplete. This is rectified in the present note. The special cases of $\mathbf{Sb}_2$ taken by Heath are:

(i) $A = B \vdash \mathrm{S}A = \mathrm{S}B$
(ii) $A = B \vdash x + A = x + B$
(iii) $A = B \vdash A + x = B + x$
(iv) $A = B \vdash x \dot{-} A = x \dot{-} B$
(v) $A = B \vdash A \dot{-} x = B \dot{-} x$

Remark In fact either (ii) or (iii) can be omitted since $x + y = y + x$ can be proved without using (ii) or (iii) and then one can be derived from the other.

In order to derive the full $\mathbf{Sb}_2$, i.e., $A = B \vdash f(A) = f(B)$, for any primitive recursive function f, it is necessary to show that the substitution theorem, $x = y \rightarrow f(x) = f(y)$, persists under definition by a primitive recursive schema. Heath shows that it persists under the recursion without parameter, which I shall call $\mathbf{R}$,

$$f(0) = (0),$$
$$f(\mathrm{S}x) = g(x, f(x)),$$

i.e., that from $x = y \;\&\; w = z \rightarrow g(x, w) = g(y, z)$ we can deduce $x = y \rightarrow f(x) = f(y)$. He then quotes a theorem of R. M. Robinson that all primitive recursive functions are generated from $0, x, \mathrm{S}x, x + y$ and $x \dot{-} y$ by substitution and the recursion $\mathbf{R}$. To complete the proof it would be sufficient to show that Robinson's reduction of primitive recursion can be carried out in the restricted primitive recursive arithmetic (i.e., without full $\mathbf{Sb}_2$). This would involve defining the pairing functions $\mathrm{J}(x, y)$, $\mathrm{K}(x)$ and $\mathrm{L}(x)$ given by Robinson, deriving their main properties, e.g. $\mathrm{L}(\mathrm{S}x) \neq 0 \rightarrow \mathrm{K}(\mathrm{S}x) = \mathrm{K}(x) \;\&\; \mathrm{L}(\mathrm{S}x) = \mathrm{S}(\mathrm{L}x)$, and checking that the substitution theorem is satisfied by them. This part was omitted by Heath, and it is not clear that this programme could be carried out.

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