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A SET-THEORETIC MODEL FOR NONASSOCIATIVE NUMBER THEORY

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1 *Introduction.* To our knowledge, the first reference to nonassociative numbers as an independent concept is in a paper of Etherington [3], in which it is related to some situations in biology. Recently, it has been shown in [1] and [7] that a suitable representation of nonassociative numbers can be a useful tool to solve some problems of coherence in the sense of [8]. Moreover, the set of nonassociative numbers is one of the simplest free algebras and can be used to give descriptions of nonassociative free algebras.

Formally, the theory **N** of nonassociative numbers bear similarities to those of the theory of natural numbers. In [4], Evans characterized the nonassociative numbers by a set of "Peano-like" axioms. In [2], these axioms were formalized and following a suggestion of Evans, it was shown that **N** is incomplete and furthermore that it is essentially undecidable. It is natural to ask if and how **N** can be formalized within formal set theory, say Zermelo-Fraenkel (ZF). In the present work we do exactly this. Furthermore, by considering variations of this model, we show that the axioms of **N** are independent.

The representations of **N** by coordinates in [1] and [7] offer the possibility of constructing other models for **N**, but they would be more complicated than ours. In this connection we refer to Freyd's Adjoint Theorem [5], one of whose consequences is the existence of free algebras, which therefore also gives a way to construct a model for **N**, but this too would be quite sophisticated.

2 *A model for nonassociative number theory.* In [2], nonassociative number theory is defined to be the first-order theory with equality, **N**, having one individual constant 1, three binary function letters corresponding to addition (+), multiplication (·), and exponentiation and whose proper axioms are:

(N1) $x_1 + x_2 \neq 1$

(N2) $x_1 + x_2 = x_3 + x_4 \,.\supset.\, x_1 = x_3 \wedge x_2 = x_4$

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