Notre Dame Journal of Formal Logic
Volume XIV, Number 1, January 1973
NDJFAM

DIAGONALIZATION AND THE RECURSION THEOREM

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In 1938 Kleene showed that if f is a recursive function then, for some number c, $\varphi_c \simeq \varphi_{f(c)}$, where φ_e is the partial recursive function with index e. Since that time other fixed-point theorems have been found with similar proofs. All of these theorems tend to strain one's intuition; in fact, many people find them almost paradoxical. The most popular proofs of these theorems only serve to aggravate the situation because they are completely unmotivated, seem to depend upon a low combinatorial trick, and are so barbarically short as to be nearly incapable of rational analysis. It is our intention, one, to put Kleene's proof on classically intuitive grounds by explaining how it can be viewed as a natural modification of an ordinary diagonal argument and, two, to present a formulation of Kleene's theorem sufficiently abstract to yield all known similar theorems as corollaries.*

In a typical diagonal argument one has a class of sequences (with terms from a set S), which he arranges as the rows of a square matrix, and a mapping α of S into S. This mapping induces an operation α^* on the class of arbitrary sequences of elements of S in the natural way—if $\langle s(i) : i \epsilon I\rangle$ is such a sequence then $\alpha^*(\langle s(i) : i \epsilon I\rangle) = \langle \alpha(s(i)) : i \epsilon I\rangle$. One then applies α^* to the sequence of diagonal elements of the matrix and shows that the resulting sequence is not a row of the matrix, thus diagonalizing himself out of the class of sequences he began with. A good example is the matrix whose rows are all infinite periodic sequences of 0's and 1's (binary expansions of rationals) with the mapping $\alpha(0) = 1$, $\alpha(1) = 0$.

Usually, as in the example just given, the rows of the matrix are closed under the operation α^*. Hence, if the diagonalization succeeds, it is usually true that the diagonal sequence itself is not one of the rows. But what if the diagonalization fails, that is, what if the diagonal sequence is one of the rows? Then the image of the diagonal sequence under α^* will also be one of the rows, which means some member of the diagonal sequence must be left unchanged under the action of α^*. In other words, α has a fixed point!

To understand Kleene's theorem in these terms, first assume f is a

*Partially supported by NSF grant GP-6897.

Received July 20, 1970, Revised April 1, 1972